THE FRANK NELSON COLE PRIZE IN ALGEBRA

The first award of the Cole prize will be made at the end of 1927 for a notable contribution to the theory of linear algebras by a resident of the United States or Canada.

1. The contribution may be either an unpublished manuscript or a paper first published during 1925, 1926, or 1927.

2. Two copies of the manuscript or printed paper should be in the hands of the Secretary of the Society, 501 West 116th Street, New York City, on or before October 31, 1927.

3. The first award will be two hundred dollars ($200,00).

4. The prize will be withheld in the absence of a sufficiently meritorious memoir.

Without excluding others, the following topics are suggested as suitable ones:

a. Determination of all division algebras of rank 5. (For those of rank 4, see F. Cecioni, PALERMO RENDICONTI, vol. 47 [1923], pages 209–254.)

b. Direct proof, without the theory of the rank equation, that an algebra whose units form a group is semi-simple.

c. Classification of nilpotent algebras.

d. Every algebra A is the sum of a semi-simple sub-algebra S and the maximal nilpotent invariant sub-algebra N. Investigate the multiplicative relations between units of S and units of N.

e. Develop a rational theory of algebras A over any field by starting with the rank function of the general element of A expressed in terms of the rank functions of the simple components of S in problem d.

f. Extend the theory of arithmetics of algebras. In particular, invent a theory of ideals for division algebras.

The literature is accessible in G. Scorza's *Corpi Numerici e Algebre,* Messina, 1921, and Dickson's *Algebras and their Arithmetics,* University of Chicago Press, 1923.

H. S. WHITE,
For the Committee of the Council.

THE PAN-AMERICAN SCIENTIFIC CONGRESS

The third Pan-American Scientific Congress was held in Lima, Peru, from December 20, 1924 to January 6, 1925. It was divided into nine section: 1. Anthropology and History; 2. Physical and Mathematical Sciences (including Geology and Geography); 3. Mining and Metallurgy; 4. Engineering; 5. Medicine and Sanitation; 6. Biology and Agriculture; 7. Law; 8. Economics and Sociology; 9. Education.

At the impressive opening ceremony, the Congress was formally welcomed by President Leguia of the Peruvian Republic and Dr. Salamón, Minister of Foreign Affairs. All the republics in the western hemisphere except Chili were represented. There were 158 members from Peru and 151 from foreign countries, including 30 from the United States, 25 from Argentina, 15 from Cuba, 9 from Venezuela, 8 from Colombia, 7 from Uruguay, 6 from Brazil, 5 from Ecuador, 5 from Mexico, etc. It was decided to hold the next Pan-American Scientific Congress in San José, Costa Rica, in 1929.

The subject of Pure Mathematics was assigned to Subsection 1 under Section 2. This subsection held one meeting for the reading of papers on December 28, 1924. About fifteen members were present, and the following three papers were read.

1. *Conjugate ordinates and their geometric applications,* by Dr. Florencio D. Jaime, President of the recently organized Argentinian Mathematical Society, and delegate from the Ministry of Public Instruction of Argentina.

2. *Elementary types of order,* by Dr. Edward V. Huntington, vicepresident of the American Mathematical Society, and delegate from the American Mathematical Society, the Mathematical Association of America, and the American Academy of Arts and Sciences.

3. *On the descriptive geometry of the sphere,* by Dr. Alejandro Guevara, Honorary Professor of the School of Engineers in Lima.

E. V. HUNTINGTON.

THE FEBRUARY MEETING IN NEW YORK

The two hundred fortieth regular meeting of the Society was held at Columbia University, on Saturday, February 28, 1925, extending through the usual morning and afternoon sessions. The attendance included the following sixty-one members of the Society:

Alexander, R. L. Anderson, C. R. Ballantine, J. P. Ballantine, Birkhoff, Blake, Bowden, R. W. Burgess, G. A. Campbell, Alonzo Church, Cole, Dadourian, Eisenhart, Eshleman, Fite, Frink, Fry, Gafafer, Gehman, Gronwall, C. C. Grove, Guggenbühl, Hausle, Hille, Himwich, Louis Ingold, Joffe, Kasner, O. D. Kellogg, Kline, Lamson, Langer, Langman, Lefschetz, Harry Levy, McGiffert, McMackin, MacColl, MacNeish, Meder, Mirick, Molina, C. L. E. Moore, Mullins, Olson, Pell, Pfeiffer, R. G. Putnam, Rainich, Reddick, Ritt, Schelkunoff, Seely, Siceloff, J. H. Taylor, H. D. Thompson, Veblen, H. E. Webb, Wedderburn, Whittemore, Wiener.

There was no meeting of the Council or of the Trustees.

At the beginning of the afternoon session, a paper was read by Professor J. W. Alexander, at the request of the Program Committee, on *Problems in the topological theory of manifolds.*

Professor Béla de Kerékjártó, of the University of Szeged, Hungary, spoke briefly of the mathematical journal recently founded at that university, the ACTA LITTERARUM AC SCIENTIARUM REGIAE UNIVERSITATIS HUNGARICAE FRANCISCO-JOSEPHINAE, SECTIO MATHEMATICARUM; the articles in this journal are in French or German. He suggested that members of the Society might call the attention of their university libraries to this publication.

President Birkhoff presided at both sessions.

Titles and abstracts of the papers read at this meeting follow below. Professor Alexander's second paper and the papers of Bennett, Dodd, Franklin, Garabedian, James, Jeffery, Libman, Osgood, Reynolds, Thomas, and Weisner were read by title; Professor Morley's paper was communicated by Mr. Rainich. Professor de Kerékjártó was introduced by President Birkhoff.

1. Professor C. L. E. Moore: *Minimal varieties of two or three dimensions whose element of arc is a perfect square.*

After determining the euclidean space of lowest dimension in which minimal varieties of two or three dimensions whose element of arc is a perfect square can lie, the author discusses the intrinsic geometry of these varieties.

2. Professor L. P. Eisenhart: *Fields of parallel vectors in a Riemannian geometry.*

In accordance with the definition introduced by Levi-Civita, parallelism of vectors depends upon the metric of the space and upon the curve along which a vector is displaced so as to remain parallel to its initial position. In a general Riemannian space there does not exist a field of vectors which are parallel to one another regardless of the curves of displacement. However, there are certain Riemannian spaces in which there are one, or more, fields of vectors possessing this property. In this paper canonical forms of the metric of these spaces are determined.

3. Mr. G. Y. Rainich: *On the Riemann tensor.*

In a flat space, to every contour corresponds its *tensor area*, an alternating tensor of rank two. If a curved space $\sum$ is immersed in our flat space, we consider for every contour on $\sum$ its *contour variation of normals* (normal flat spaces). For infinitesimal contours in the vicinity of a given point, the contour variation of normals is a linear function of the tensor area, and this function is the *Riemann tensor* at the point considered. By integrating (in a sense previously introduced) the Riemann tensor over a surface, we obtain again the finite variation of normals corresponding to the contour limiting this surface. If the curved space is four-dimensional, the Riemann tensor is the sum of two parts, of which the first assigns to absolutely perpendicular planes equal curvatures and the second opposite curvatures. The first part involves 11 and the second 9 constants. The second part has the same coefficients as the "cosmological energy tensor"; its vanishing is equivalent to the cosmological equations. In physical space-time, the second part depends only on 5 constants and can be expressed quadratically through the electromagnetic tensor.

4. Mr. G. Y. Rainich: *Integrals in curved space.*

It has been previously shown how integrals can be formed when a tensor field has superfluous indices; this required the consideration of the containing flat space. However, when the tensor field depends on the superfluous indices in the same way as on the others, it is possible to form integral expressions without leaving the curved space, by presenting the tensor as the sum of products of tensors of which integrals can be taken and by forming the sum of the products of these integrals. This is first applied to the case of the indicator of a surface; it is shown that only for a minimal surface do the expressions considered vanish over a reducible contour. For an irreducible contour on a one-sided surface they give rise to residues; the residues are the same for all simple irreducible contours of a given surface. In the case of the physical space-time, the second part of the Riemann tensor when treated in a similar manner gives an expression which vanishes on a reducible closed surface, and for a surface surrounding a singularity it gives the square of the electric charge.

5. Professor Frank Morley: *Comitants of a curve under inversion.*

This paper deals with the comitants under inversion of a plane curve. The equation of such a curve, expressed in circular coordinates, is obtained by bordering a self-conjugate or hermitian matrix. The determinant of this matrix is an invariant. If with the curve we combine the special curve which represents a repeated point, the determinant of the pencil gives the reciprocal curve of the given one. The pencil formed by a curve and its reciprocal gives a series of invariants. The results are included in the memoir of Kasner (TRANSACTIONS OF THIS SOCIETY, vol. 1) but it may be of interest to consider this important question from all points of view.

6. Professor Edward Kasner: *Null geometry.*

A null manifold (all distances vanishing) of k dimensions cannot exist in a euclidean space of less than $2k$ dimensions. In space of $2k$ dimensions, null k-flats exist, and in higher spaces curved null spaces exist.

7. Professor Louis Ingold: *Extensions of the equations of Gauss and Codazzi.*

The paper considers Riemann spaces R_n determined by a general fundamental vector $F(u^1, u^2, \cdots, u^n)$. The first partial derivatives $\partial f/\partial u^i = f_i$ are regarded as independent tangent vectors to R_n. The first fundamental quantities E_{ij} are the scalar products $f_i f_j$. Second fundamental quantities L_{ijrs} are defined as the scalar products $N_{ij} N_{rs}$ of vectors normal to R_n, where N_{ij} is linear in the corresponding f_{ij} and the tangent vectors f_k. Third fundamental quantities are defined in a similar way from normals N_{ijk} linear in f_{ijk} and the normals and tangents defined above. These are orthogonal to the N_{rs}. In case the vector f satisfies a completely integrable system of $n-1$ linear differential equations of the second order, all of the normals N_{ij} are expressible in terms of a single normal N; and the first and second fundamental quantities satisfy the usual equations of Gauss and Codazzi. If it is assumed that this is not true, but that the normals N_{ijk} are expressible in terms of a single normal vector, then analogous necessary relations connecting the first, second, and third fundamental quantities are obtained. Further extensions are obvious.

8. Dr. Harry Levy: *Tensors determined by a hypersurface in a Riemann space.*

Consider a hypersurface, V_n, of n dimensions, embedded in a general Riemann space of $n+1$ dimensions. If we let V_n be represented by the equation $u^0 = 0$ and if we take for the parameter u^0 the arc of the geodesics normal to V_n, the linear element of the space becomes $ds^2 = (du^0)^2 + C_{\alpha\beta} du^\alpha du^\beta$ $(\alpha, \beta = 1, 2, \cdots, n)$. We find that the coefficients in the expansion of $C_{\alpha\beta}$ in a power series in u^0 are a sequence of tensors intimately connected with the geometry of the hypersurface. In the first part of this paper we deduce a number of identities which hold in general tensor analysis. In the second part we consider these tensors and obtain general properties of Riemann space.

9. Dr. Harry Levy: *Symmetric tensors of the second order whose covariant derivatives vanish.*

We give a complete geometric characterization of tensors of the kind mentioned in the title, showing that necessary

and sufficient conditions that the covariant derivative of b_{ij} vanish are that b_{ij} be a first integral of the differential equations of the geodesics of the space, and that the space whose fundamental form is $b_{ij}\,dx^i\,dx^j$ admit of geodesic representation upon the given space. We prove further that, in a space of constant curvature, a tensor whose covariant derivatives are zero is a constant multiple of the fundamental tensor.

10. Professor H. L. Olson: *Congruences with constant absolute invariants.*

This paper is based on the researches of Wilczynski, chiefly on his memoir *Sur la théorie générale des congruences*, published by the Royal Academy of Belgium. Regarding the special type of congruences considered, it is shown that both sheets of the focal surface have constant absolute invariants, and that the cuspidal edges of the developables are two families of projectively equivalent anharmonic curves. The Laplace transforms of any such congruence have constant absolute invariants. Necessary and sufficient conditions that a congruence be projective to its 1st or (—1)st Laplace transform are that its absolute invariants be constants and that $\mathfrak{B}^{(x)} - \mathfrak{B}^{(y)}$ and $\mathfrak{C}''^{(y)} - \mathfrak{C}''^{(x)}$, defined by Wilczynski, vanish identically. Any congruence of the latter type is contained in a tetrahedral complex or one of the limiting forms of the tetrahedral complex.

11. Professor W. F. Osgood: *On normal forms of differential equations.*

Linear differential equations of the second order (and their Schwarzian resolvents) are considered on an algebraic configuration which is first taken in the form of Noether's normal curve. The latter is now projected on a pencil of hyperplanes, and the algebraic configuration is treated (i) in the binary domain thus arising; and (ii) in the domain corresponding to a uniformization by means of automorphic functions with limiting circle. The invariant, to the existence of which Klein was led in the case of the group of the binary linear transformations of Case (i), is obtained explicitly and independently in terms of the uniformizing variables of Case (ii), and its properties are read off by means of the automorphic functions.

12. Professor A. A. Bennett: *Two new arctangent relations for* π.

In this paper the author gives two new formulas for use with Gregory's series in computing π. The first is not unsuited for pencil and paper computation, and gives much more rapidly convergent series than does Machin's well known relation. The second is intended for use with a computing machine. The first may be written in the form $\pi/4 = 12 \operatorname{arcctn} 18 + 3 \operatorname{arcctn} 70 + 5 \operatorname{arcctn} 99 + 8 \operatorname{arcctn} 307$. The second has more arguments, but has 200 for its smallest argument.

13. Professor A. A. Bennett: *Diophantine arccotangent relations.*

In this paper the author shows that a necessary and sufficient condition that the equation $\operatorname{arcctn} x_1 + \operatorname{arcctn} x_2 = \operatorname{arcctn} y_1 + \operatorname{arcctn} y_2$ be satisfied by given integers is that certain corresponding expressions u_1, u_2, v_1, v_2 satisfy the congruences $u_1 v_1 \equiv u_1 v_2 \equiv u_2 v_1 \equiv u_2 v_2 \pmod{x_1 + x_2}$. Several infinite systems of solutions are obtained, and a complete table of all solutions, up to $x_1 + x_2 = 25$, is given. The simpler case of the three term arccotangent relation is disposed of. The problem in both cases reduces to a utilization of a table of numerical factors of $x^2 + 1$, and the theory of the construction and checking of such a table is developed.

14. Professor E. L. Dodd: *The fitting of curves by the use of moments and conjugate moments.*

If f is the frequency of x, the conjugate rth moment is defined as $\sum f \cdot x^r \cdot \operatorname{sign} x$, with $r = 0, 1, 2, \cdots$. The author shows that conjugate moments are especially useful in curve-fitting, since double the usual number of parameters may be introduced. Suppose, for example, that the frequency function is $y = e^{-x^2/2}(A + Bx + Cx^2 + \cdots + Kx^{2s+1})$, where the arithmetic mean has been made the origin and the standard deviation the unit. Using moments up to the sth with their conjugates, the $s+1$ alternate parameters $A, C, E, \cdots$ may be obtained by solving $s+1$ linear equations, and the remaining parameters likewise. Thus we may fit a six-parameter curve to a distribution without going higher than second moments (it seems especially advisable to avoid high moments when extreme variates are irregular) while

even twenty-two-parameter curves become tractable with the Pearson tables.* For comparison, the distribution of 1130 pensioners exhibited by Arne Fisher† was fitted by a six-parameter curve of the above form (with corrections for grouping) and 13 positive and 13 negative aberrations were obtained, the same absolute total (26) as was obtained by Fisher by the Charlier method involving moments up to the fourth.

15. Dr. E. E. Libman: *Linear complex of conics.*

A conic in space involves in its specification eight arbitrary constants. When these are expressed as integral linear non-homogeneous functions of three independent parameters, the system so obtained is called a linear complex. The linear complex of conics so defined has the following properties: (I) There is one and only one conic upon each plane in space. (II) The centers of the degenerate conics of the complex lie upon a quadric surface and their planes envelope another quadric surface; these are called the singular quadrics of the complex. (III) The conics through a point are in one to one correspondence with the points of a plane quartic curve associated with the point. The plane of the quartic, and the point are called polar and pole with respect to the complex. (IV) The points that lie upon their polars are on one of the singular quadrics, while the planes that contain their poles envelope the other.

The machinery used in the investigation is quaternions with the usual Hamiltonian symbols.

16. Professor C. N. Reynolds: *On the map-coloring problem, with particular reference to connected sets of pentagons.*

In this paper the author considers the problem of developing, by analytic methods, the implications of such geometric reductions of the map-coloring problem as have been published; i. e., those due to Kempe, Birkhoff, and Franklin, but not those of Errera. His method is that of applying linear difference equations to the study of the topological properties of irreducible connected configurations of pentagons. It is then shown that the reductions considered imply the possibility of coloring, in four colors, the map

* *Tables for Statisticians and Biometricians*, Tables II and IX.

† *Frequency Curves*, 1922, p. 48.

of any spherical surface which is divided into not more than twenty-seven regions. Two maps of twenty-eight regions each which can be colored in four colors but which are irreducible, in the sense defined above, are then exhibited as a check on the closeness of fit between the analytic methods presented and the geometric problem considered.

17. Professor C. A. Garabedian: *Solution of the problem of the thick rectangular plate, clamped or supported at its edges and under uniform or central load.*

The author has already given a solution of the problem of the thick rectangular plate supported at its edges and under uniform pressure.* The method employed may be successfully applied to all four problems mentioned in the title above, and it is noteworthy that the resulting formulas of displacement are in each case simply expressible in terms of the solution (and its derivatives) of the corresponding thin plate problem. Since the four thin plate solutions at issue are given in complete form in Hencky's thesis,† rigorous and usable formulas for the thick plate are now available to the engineer. The solutions of the present note are analogous to the author's solutions of the same problems in circular plates,‡ and there are certain interesting remarks applicable to both rectangular and circular cases. Finally the method is capable of including plates of variable thickness. An abstract of results will be found in the COMPTES RENDUS of January 26, 1925; details will be given in an extended paper now in preparation.

18. Professor Glenn James: *A complete solution of the cubic equation.*

This paper determines an interval in which one and only one of the roots of a cubic equation lies, and then develops an interpolation theorem concerning a root in a given interval. Upon the basis of this theorem, a root is secured as a function of the coefficients which can be evaluated for numerical equations by a certain recurrence formula. The other roots are obtained in general form from the reduced equation.

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* COMPTES RENDUS, vol. 178 (1924), p. 261.

† *Der Spannungszustand in rechteckigen Platten,* Darmstadt, 1913 (published by Oldenbourg, Munich).

‡ TRANSACTIONS OF THIS SOCIETY, vol. 25 (1923), pp. 379–388.

19. Professor R. L. Jeffery: *Functions of two variables for which the double integral does not exist.*

It is pointed out by E. W. Hobson* that even though a function of two variables is continuous in each variable separately, the double integral in the sense of Riemann need not exist. This is obviously true if the function is unbounded, but there seems to be nowhere in the literature a bounded function which illustrates Hobson's observation. In this note, a function $f(x, y)$ is constructed on the unit square which is bounded and continuous in each variable separately, but which is discontinuous in the two variables with a saltus equal to unity at every point of a two-dimensional set whose plane content is $\frac{1}{4}$. Hence, for this function, the Riemann double integral fails to exist.

20. Dr. Louis Weisner: *On the number of elements in a group which have a power in a given conjugate set.*

This paper will appear in full in an early number of this BULLETIN.

21. Dr. J. M. Thomas: *The number of even and odd absolute permutations of n letters.*

This paper appears in full in the present number of this BULLETIN.

22. Dr. J. M. Thomas: *Note on the projective geometry of paths.*

The projective curvature tensor discovered by Weyl (GÖTTINGER NACHRICHTEN, 1921, p. 99) is derived in a manner analogous to the derivation of the ordinary curvature tensor, and a new projective tensor of rank 13 is obtained.

23. Professor J. W. Alexander: *Problems in the topological theory of manifolds.*

This paper will appear in an early issue of this BULLETIN.

24. Professor J. W. Alexander: *Combinatorial analysis situs.*

The author determines a necessary and sufficient condition for the homeomorphism of two complexes which is expressed in terms of cells and their incidence relations rather than

* *Theory of Functions of a Real Variable*, 2d edition, vol. 1, § 365.

in terms of points and limit points. He is then able to develop the theory of the connectivity numbers and intersection invariants without further appeal to continuity considerations.

25. Professor O. D. Kellogg: *Note on the convergence of real power series representing harmonic functions.*

If a series of spherical harmonics in x, y, and z converges within the sphere $x^2+y^2+z^2<R^2$, then the power series in x, y, and z which represents the same function converges within the octahedron $|x|+|y|+|z|<R$. A corresponding result holds in the case of n variables. For $n=2$, the complete two-dimensional region of convergence is thus characterized, as has been shown by Bôcher. For $n>2$, however, the complete n-dimensional region is not, in general, given by the theorem indicated above.

26. Professor J. F. Ritt: *Transcendental transcendency of certain functions of Poincaré.*

Poincaré, in 1890, established the existence of a large class of meromorphic functions $y(x)$ for which an $m \neq 1$ exists such that $y(mx)$ is a rational function of $y(x)$. The functions e^x, $\cos x$, and $\wp(x)$ belong to this class. In the present paper, it is proved that with the exception of the three functions just named and certain others closely related to them, the functions of Poincaré are transcendentally transcendental, that is, they do not satisfy algebraic differential equations.

27. Professor J. R. Kline: *Concerning the sum of a countable infinity of mutually exclusive continua.*

Sierpinski has proved that no bounded closed connected set can be the sum of a countable infinity of mutually exclusive closed connected sets. In her dissertation, Miss Anna Mullikin gave an example of a connected set which is the sum of a countable infinity of mutually exclusive closed connected sets. In the present paper it is proved that if S is a point set which is the sum of a countable infinity $S_1, S_2, \cdots$ of closed connected sets having the property that given any ϵ, greater than zero, there are at most a finite number of the sets $S_1, S_2, \cdots$ of diameter greater than ϵ, then S is not connected.

28. Dr. H. M. Gehman: *On extending a continuous* (1,1) *correspondence of two plane continuous curves to a correspondence of their planes.*

It is proved that if any two plane continuous curves are in continuous (1,1) correspondence in such a way that sides of arcs are preserved under the correspondence, then a continuous (1,1) correspondence of their planes can be defined in such a way that the correspondence of the two continuous curves is preserved. This generalizes a theorem previously announced for the case of continuous curves that contain no simple closed curves.

29. Professor S. Lefschetz: *On the problem of inversion of abelian integrals.*

This paper gives a proof of the existence and uniqueness of the solution of Jacobi's inversion problem by means of the implicit function theorem.

30. Dr. Philip Franklin: *Osculating curves and surfaces.*

In this paper several conditions are given under which a curve or surface, obtained as the limit of a sequence of curves or surfaces having a fixed number of points in common with a given curve or surface, osculates this given one. As a typical theorem we have the following: If in some neighborhood of a point P on a curve its defining function possesses a continuous fourth derivative, and if at this point there is a unique osculating conic ($y''' \neq 0$) a sequence of conics having five points in common with the given curve approaches the osculating conic at P when the five points close down on P.

31. Professor R. E. Langer: *On the momental constants of a summable function.*

This paper deals with the momental constants defined by Haskins for any summable function. A set of necessary conditions on the constants of any enumerably infinite set if they are to be the momental constants of a summable function with finite measurable bounds is deduced. These conditions are then shown to be sufficient as well. The determination of the function having a given set of momental constants is made to depend upon the solution of an associated "problem of moments".

32. Professor Norbert Wiener: *The solution of a difference equation by trigonometric integrals.*

In every case in which Nörlund has solved the equation $f(x+1)-f(x)=\varphi(x)$ in real variables, his principal solution may be resolved into the sum of two functions $f_1(x)$ and $f_2(x)$, such that $f_1(x)=-\sum_{s=0}^{\infty}\varphi_1(x+s)$ and $f_2(x)=\int_a^x \varphi_2(z)\,dz+\sum_{\nu=1}^{\infty}(B_\nu/\nu!)\,\varphi_2^{(\nu-1)}(x)$, where $\varphi(x)=\varphi_1(x)+\varphi_2(x)$.

33. Dr. T. H. Gronwall: *On Gibbs' phenomenon.*

This paper investigates the condition for the occurrence of a Gibbs' phenomenon in the Cesàro sums of order k $(0\leqq k\leqq 1)$ of a Fourier series.

34. Professor Einar Hille: *Some remarks on Dirichlet's series.*

This paper is concerned with certain relations between Dirichlet's, factorial, and binomial series. The starting point is a paper by Wigert (ARKIV FÖR MATEMATIK, ASTRONOMI, OCH FYSIK, vol. 7 (1911), No. 26) in which a binomial series is expressed as the sum of an entire function and a factorial series. The corresponding relation for Dirichlet's series can be obtained from Cahen's expression for such a series by means of a Laplace integral. The desired relation is obtained by applying Pincherle's results on *fonctions déterminates* to this integral. It is observed that Cahen's integral can be used for the purpose of summing a certain class of everywhere divergent Dirichlet's series. Some theorems on the analytic nature of the functions represented by certain Dirichlet's series are given as an application. These results were first derived by Hardy using Borel's method of summation. Various generalizations are considered.

35. Professor Béla de Kerékjártó: *Remarks on convex regions.*

The author discusses the theorems of E. Helly concerning common points of convex regions and of simply connected plane regions.

W. B. FITE,
Acting Secretary.

THE NUMBER OF EVEN AND ODD ABSOLUTE PERMUTATIONS OF n LETTERS*

BY J. M. THOMAS†

A permutation of n letters is called absolute if it leaves no letter fixed. A formula for the total number of such permutations is well known, but no separate formulas for the numbers of even and odd absolute permutations seem to be given in the literature. It is the purpose of this note to show how such formulas can be derived.

Consider the determinant of order n

$$D = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 0 \end{vmatrix},$$

all of whose elements are equal to 1, except those on the main diagonal, which equal 0. By writing it in the form

$$\begin{vmatrix} 1-1, & 1+0, & 1+0, & \cdots, & 1+0 \\ 1+0, & 1-1, & 1+0, & \cdots, & 1+0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1+0, & 1+0, & 1+0, & \cdots, & 1-1 \end{vmatrix},$$

we see that it is expressible as the sum of 2^n determinants which fall into three classes. Those determinants having more than one column of 1's are zero. Those having exactly one column of 1's are n in number, and their common value is $(-1)^{n-1}$. Finally there is just one determinant containing no column of 1's, and its value is $(-1)^n$. Hence the value of D is $(-1)^{n-1}(n-1)$.

On the other hand if we consider the expanded form of D, say

$$D = \sum \pm a_{1i}a_{2j} \cdots a_{nk},$$

* Presented to the Society, February 28, 1925.

† National Research Fellow in Mathematics.

we see that any term which does not correspond to an absolute permutation of the n indices $i, j, \cdots, k$ contains an element from the main diagonal and is therefore zero. The value of a term corresponding to an absolute permutation is $+1$ or -1 according as the permutation is even or odd. The value of D is therefore the difference between the number of even absolute permutations, N_e, and the number of odd absolute permutations, N_o, that is

$$N_e - N_o = (-1)^{n-1}(n-1). \tag{1}$$

Moreover we have*

$$N_e + N_o = n! \sum_{r=2}^{n} \frac{(-1)^r}{r!}. \tag{2}$$

From (1) and (2) N_e and N_o can be calculated.

PRINCETON UNIVERSITY

THE ABSOLUTE VALUE OF THE PRODUCT OF TWO MATRICES†

BY J. H. M. WEDDERBURN

1. *Introduction.* If $a = (a_{pq})$ is a matrix of order n whose elements are ordinary complex numbers, the absolute value of a is defined as $\sqrt{\sum a_{pq}\bar{a}_{pq}}$, where $\bar{a} = (\bar{a}_{pq})$ is the matrix whose coefficients are the conjugates of the corresponding coefficients in a; we shall denote it here by $|a|$, a special symbol being convenient since the absolute value of a scalar matrix λ is not $|\lambda| \equiv \text{mod } \lambda$ but $n^{1/2}|\lambda|$. This definition has been freely used by writers on differential equations; but, in spite of this, its properties with regard to multiplication have seemingly escaped notice, or are at least not well known.

* Cf. Seelhoff, ARCHIV DER MATHEMATIK UND PHYSIK, (2), vol. 1, p. 100.

† Presented to the Society, May 2, 1925.

The principal properties of $|a|$ are as follows:

$$(1) \qquad |a+b| \leqq |a|+|b|,$$

$$(2) \qquad |\lambda| = n^{1/2}|\lambda|, \quad (3) \qquad |\lambda a| = |\lambda||a|, \qquad (\lambda \text{ scalar})$$

$$(4) \qquad |ab| \leqq |a||b|.$$

Relation (4), which is the only one which is not obvious, follows immediately from the identity

$$|a|^2|b|^2 = \sum_{pr} a_{pr}\bar{a}_{pr} \sum_{qs} b_{sq}\bar{b}_{sq}$$

$$= \sum_{pq}\left\{\frac{1}{2}\sum_{rs}(a_{pr}\bar{b}_{sq}-a_{ps}\bar{b}_{rq})(\bar{a}_{pr}b_{sq}-\bar{a}_{ps}b_{rq})+\sum_{r} a_{pr}b_{rq}\sum_{s}\bar{a}_{ps}\bar{b}_{sq}\right\},$$

whence, seeing that the terms of the first summation within the braces are real and positive, we have

$$|a|^2|b|^2 \geqq \sum_{pq}\sum_{r} a_{pr}b_{rq}\sum_{s}\bar{a}_{ps}\bar{b}_{sq} = |ab|^2.$$

The equality sign holds in (4) only when a_{rs} and b_{rs} have the form $a_{rs} = \lambda_r\mu_s$, $b_{rs} = \bar{\mu}_r\nu_s$.

Similar matrices do not in general have the same absolute value; $|bab^{-1}|$ and $|a|$ are the same however if $b = uc$, where $ca = ac$ and u is a unitary matrix.*

2. *Negative Powers of a Matrix.* As pointed out by Schur, $|a|^2$ may be defined as the trace of $\bar{a}'a$, which is a positive Hermitian matrix and therefore has positive real roots, say $g_1, g_2, \cdots, g_n$. If

$$s_r = \sum g_1^r,\ p_r = \sum g_1 g_2 \cdots g_r,\ s \equiv s_1 = |a|^2,\ p \equiv p_n = |\det a|^2,$$

* Cf. Schur, *Über die characteristischen Wurzeln einer linearen Substitution*, MATHEMATISCHE ANNALEN, vol. 66 (1909), pp. 488–510, pp. 490, 492. Since replacing a matrix by a similar one corresponds to a change of coordinates when the matrix is regarded as a linear transformation, it follows from Schur's work that, when the elementary divisors of a are simple, $|a|$ has its minimum value when a is represented as a diagonal matrix, that is, in its normal form; and it seems probable that the normal form also gives the minimum value even if the elementary divisors are not simple.

then it follows readily from the inequality connecting the arithmetic and geometric means that*

$$r^r p_r \leqq \sum(g_1+g_2+\cdots+g_r)^r \leqq \binom{n-1}{r-1} s^r,$$

$$p_r \geqq \binom{n}{r} p^{\binom{n-1}{r-1}/\binom{n}{r}} = \binom{n}{r} p^{r/n},$$

whence

$$\binom{n}{r} p^{r/n} \leqq p_r \leqq r^{-r}\binom{n-1}{r-1} s^r. \tag{5}$$

If $C_r(a)$ is the rth compound of a,† $p_r = |C_r(a)|^2$; hence, if we put α for the determinant of a, (5) becomes

$$\binom{n}{r} |\alpha|^{2r/n} \leqq |C_r(a)|^2 \leqq r^{-r}\binom{n-1}{r-1} |a|^{2r}. \tag{6}$$

Now $a^{-1} = C_{n-1}(a)/\alpha$, provided $\alpha \neq 0$; hence

$$n^{1/2}|\alpha|^{-1/n} \leqq |a^{-1}| \leqq (n-1)^{-\frac{(n-4)}{2}} |a|^{n-1}/|\alpha|. \tag{7}$$

This inequality enables us to deal with expressions containing negative powers of a matrix. The following inequality is, however, sometimes more convenient. Since $aa^{-1} = 1$, it follows from (4) that

$$|a^{-1}| \leqq \frac{n^{1/2}}{|a|}, \tag{8}$$

and, since‡ $a = e^{\log a}$, $a^{-1} = e^{-\log a}$, both $|a|$ and $|a^{-1}|$ are less than or equal to $e^{|\log a|}$.

* For $r = n$ this inequality, which is an extension of Hadamard's expression for the maximum of the absolute value of a determinant, is given (loc. cit., p. 496) by Schur, who also derives a number of other interesting inequalities.

† It must be remembered here that the order of $C_r(a)$ is $\binom{n}{r}$, so that, if for example $a = 1$, then $|C_r(a)|^2 = \binom{n}{r}$ and not n. References to the literature of compound matrices will be found in Pascal, REPERTORIUM DER HÖHEREN MATHEMATIK, vol. 1, pp. 138–146.

‡ An elementary discussion of the logarithm of a matrix is given by the author, *The automorphic transformation of a bilinear form*, ANNALS OF MATHEMATICS, (2), vol. 23 (1921), pp. 122–125.

3. *Infinite Series and Products of Matrices.* The use of (4) makes the discussion of infinite series and products of matrices somewhat more compact than is the case with the usual methods.* We shall give here only a few elementary illustrations.

If a_r $(r = 0, 1, 2, \cdots)$ is an infinite sequence of matrices and x a variable matrix, and we set $|a_r| = \alpha_r$, $|x| = \xi$, then $\sum a_r x^r$ converges absolutely if $\sum |a_r x^r|$ converges, and hence also when $\sum \alpha_r \xi^r$ is convergent. In particular, $e^x = \sum_0^\infty x^r/r!$ converges for every matrix x and $|e^x| \leqq e^{|x|}$. Similarly $\log(1+x) = \sum_1^\infty (-1)^{r-1} x^r/r$ converges for $|x| < 1$. Here, however, the condition is much too strong; for, if $x = 1/3$, say, then $|x| = n^{1/2}/3$, which is greater than 1 if $n > 9$, while the series converges absolutely in the ordinary sense. A better form of the condition in this and similar cases is $|xc|/|c| < 1$, where c is some non-singular matrix. Any criterion of this sort has distinct limitations; for it is not difficult to show that all that is necessary for the convergence of $\sum a_r x^r$ is that the series converge when x is replaced by the ordinary absolute value of the root of x of greatest absolute value.

4. *Examples.* The following examples† illustrate the application to infinite products. Let c be an arbitrary matrix, and set

$$(9) \qquad \begin{aligned} P_m &= (1+a_1)(1+a_2)\cdots(1+a_m), \\ Q_m &= (1+|a_1|)(1+|a_2|)\cdots(1+|a_m|); \end{aligned}$$

then

$$(10) \qquad |P_m - 1| \leqq Q_m - 1 < e^{\sum |a_r|} - 1,$$

* For applications to differential equations, for example, reference may be made to Schlesinger, *Vorlesungen über lineare Differentialgleichungen*, Leipzig, 1908, pp. 86–90, 41–42.

† The inequalities given here are those required in proving the existence of a solution of a system of linear differential equations by the method of successive approximations; cf. Schlesinger, loc. cit., pp. 3–16.

$$|P_m c| \leqq e^{\sum |a_r|} |c|, \tag{11}$$

$$|P_m - P_k| \leqq Q_m - Q_k \leqq \sum_k^m |a_r| e^{\sum_1^m |a_r|}. \tag{12}$$

It is sufficient here to give the proofs of (10) and (11). Expanding (9), we have

$$P_m = 1 + \sum_p a_p + \sum_{p<q} a_p a_q + \sum_{p<q<r} a_p a_q a_r + \cdots,$$

therefore

$$|P_m - 1| \leqq \sum_p |a_p| + \sum_{p<q} |a_p||a_q| + \cdots = Q_m - 1,$$

$$|P_m c| \leqq (1 + \sum_p |a_p| + \sum_{p<q} |a_p||a_q| + \cdots)|c| = Q_m |c|.$$

As another example we may take the proof of the existence of a solution of the differential equation

$$\frac{dx}{dt} = ax, \tag{13}$$

where $a = a_0 + a_1 t + a_2 t^2 + \cdots$ is a matrix which is analytic in the scalar variable t at $t = 0$. Let C be the circle of convergence of the series for a, that is, if $|a_r| = \alpha_r$; then $\alpha(t) = \sum \alpha_r t^r$ converges in C; and set $x = \sum x_r t^r$ in (13). Comparing like powers of t we get

$$s x_s = a_0 x_{s-1} + a_1 x_{s-2} + \cdots + a_{s-1} x_0,$$

so that $s|x_s| \leqq \alpha_0 |x_{s-1}| + \alpha_1 |x_{s-2}| + \cdots + \alpha^{s-1} |x_0|$. Let $\xi_0 = |x_0|$, $\xi(t) = \xi_0 e^{\int_0^t \alpha(t)\,dt} = \sum \xi_r t^r$, then $d\xi/dt = \alpha(t)\xi$, so that $s\xi_s = \alpha_0 \xi_{s-1} + \alpha_1 \xi_{s-2} + \cdots + \alpha_{s-1} \xi_0$. It follows that $|x_s| \leqq \xi_s$; for if this is true for $s = 1, 2, \cdots, i-1$, then

$$i|x_i| \leqq \sum_j \alpha_j |x_{i-j-1}| \leqq \sum_j \alpha_j \xi_{i-j-1} = i\xi_i.$$

The series for x therefore also converges in C and $x = x_0$ when $t = 0$.

PRINCETON UNIVERSITY

ON THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS A SUM OR DIFFERENCE OF TWO CUBES*

BY E. T. BELL

1. *Introduction and Summary.* Let $C(n)$ denote the number of integer solutions $(x, y) = (\xi, \eta)$, $\xi > 0$, $\eta > 0$, of $x^3 + y^3 = n$, the pair (ξ, η), (η, ξ) being considered as a single solution, and $D(n)$ the number of integer solutions (x, y), $x > 0$, $y > 0$ of $x^3 - y^3 = n$, $n > 0$. If the pair (ξ, η), (η, ξ) in $C(n)$ be counted as two solutions, the total number is evidently $2C(n)$ or $2C(n) - 1$ according as n is not or is the double of an integer cube > 0. No determination of $C(n)$, $D(n)$ seems to have been made. It will be of interest therefore to record forms of these functions *depending only upon the real divisors of* n, in analogy to the classical results for $x^2 \pm y^2 = n$. These forms also indicate fairly expeditious means for finding all the resolutions of n into a sum or difference of two cubes.

We denote by $\psi(z)$ the well known function whose value is 1 or 0 according as z is or is not an integer square $\geqq 0$. In the sequel only integer arguments z occur. For $S(n) \equiv C(n)$ or $D(n)$ we find the following:

(1) $n \equiv 0 \bmod 3$, $\quad n \not\equiv 0 \bmod 9$, $\quad S(n) = 0$.

(2) $\quad n \equiv 0 \bmod 9$, $\quad S(n) = \sum \psi(4t - 3d^2)$,

the $\sum$ extending to all pairs (t, d) of conjugate divisors of $n/9$ such that

$$\frac{1}{3}\sqrt[3]{n} < d \leqq \frac{1}{3}\sqrt[3]{4n}, \text{ or } d < \frac{1}{3}\sqrt[3]{n},$$

according as $S = C$ or D.

(3) $\quad n \equiv \pm 1 \bmod 3$, $\quad S(n) = \sum \psi\left(\frac{4t - 3d^2}{3}\right)$,

* Presented to the Society, San Francisco Section, June 19, 1925.

the $\sum$ extending to all pairs (t, d) of conjugate divisors of n such that $d \equiv \pm 1 \bmod 3$, and

$$\sqrt[3]{n} < d \leqq \sqrt[3]{4n}, \text{ or } d < \sqrt[3]{n},$$

according as $S = C$, D, the upper or the lower sign $\pm$ being taken throughout.

Corresponding to the exact formulas of Gauss and Eisenstein in the case of two squares for the number of lattice points in the circle of radius $\sqrt{n}$ with center at the origin, we have here

$$(4) \qquad \sum_{r=1}^{n} C(r) = \frac{1}{2}\left\{[\sqrt[3]{n/2}] + \sum_{r=1}^{N}[\sqrt[3]{n-r^3}]\right\},$$

where $N = [\sqrt[3]{n}]$, and $[z]$ = the greatest integer $\leqq z$. This and the formulas for the circle are obvious cases of the following. Let $M(n)$ denote the total number of pairs of integers (x, y), $x > 0$, $y > 0$, satisfying $n = ax^r + by^s$, where a, b, r, s are integers > 0. Then

$$(5) \qquad \sum_{\alpha=1}^{n} M(\alpha) = \sum_{\alpha=1}^{N}[\sqrt[r]{(n-b\alpha^s)/a}], \quad N \equiv [\sqrt[r]{n/b}].$$

2. *Proofs of* (1)—(5). Putting

$$n = dt = (x+y)(x^2-xy+y^2) = x^3+y^3,$$

we take $x+y = d$, $\quad x^2-xy+y^2 = t$, and hence

$$3x^2-3dx+(d^2-t) = 0.$$

The necessary and sufficient condition that this equation have an integer root is $4t-d^2 = 3w^2$, w an integer. Since $4t-d^2$ and d are of like parity, so also are d, w. Provided $d > w$, the values of (x, y) are then $\frac{1}{2}(d \pm w)$ in either order, and hence from the definition of $C(n)$ we may take $w \geqq 0$, $x = \frac{1}{2}(d+w)$, $y = \frac{1}{2}(d-w)$. From $w^2 \geqq 0$, $d > w$, we have at once the first inequality stated in (3).

For a particular d there is one and only one t, and hence not more than one w; so that each d gives not more than one (x, y), and the (x, y)'s for two different d's are distinct.

The total number of pairs (x, y) is therefore equal to the number of integer squares $(4t-d^2)/3$. Of the 9 possible cases only the following give $4t-d^2 \equiv 0 \bmod 3$,

$$(n, d, t) \equiv (0, 0, 0), \quad (1, 1, 1), \quad (2, 2, 1) \bmod 3.$$

In the case (0, 0, 0) we set $(d, t) = (3d_1, 3t_1)$, and hence $n = 9n_1$ $(n_1 = d_1 t_1)$. Reduction of the conditions upon d, and final replacement of (n_1, d_1) by (n, d) with the proper qualifications gives $C(n)$ as stated in (2); the $C(n)$ of (1), (3) are equivalent to what precedes.

For $D(n)$ we proceed similarly from

$$n = dt = (x-y)(x^2+xy+y^2) = x^3-y^3,$$

setting $x-y=d$, and hence $3x^2-3dx+(d^2-t)=0$. The condition for an integer root is as before, and it is seen immediately that we can take $x = \frac{1}{2}(w+d)$, $y = \frac{1}{2}(w-d)$, $w>0$, with the condition $w-d>0$, which includes necessarily $w>0$. Hence the sole condition is $d < \sqrt[3]{n}$.

The formula (5) is either evident or it is the immediate consequence of equating coefficients of q^n in the expansions of

$$\frac{1}{1-q}\, x\{f(q^a)\ g(q^b)\}, \qquad \left\{\frac{f(q^a)}{1-q}\right\} x g(q^b),$$

where

$$f(q) = \sum_0^\infty q^{n^r}, \qquad g(q) = \sum_0^\infty q^{n^r}, \qquad |q|<1.$$

3. *Decomposition into Cubes.* The least favorable situation is that in which n is so large that its resolution into factors is impracticable. It will be sufficient to indicate the process for $C(n)$ when $n \equiv 0 \bmod 9$. We write down the $[\sqrt[3]{4n}/3] - [\sqrt[3]{n}/3]$ integers satisfying the first inequality in (2), test which of these divide $n/9$, obtaining thus the pairs (d_i, t_i) $(i = 1, 2, \cdots)$ of conjugate divisors. If of these $i = a, b, \cdots$ give all pairs such that $4t_j - 3d_j^2$ $(j = a, b, \cdots)$ is a square, all the solutions of $x^3+y^3 = n$ with $x>0$, $y>0$ are

$$x=\frac{1}{2}\left\{3d_j+\sqrt{4t_j-3d_j^2}\right\}, \qquad y=\frac{1}{2}\left\{3d_j-\sqrt{4t_j-3d_j^2}\right\}.$$

Numerous devices for shortening the computations are suggested by numerical work, whether or not the prime factor resolution of n be feasible.

As an immediate consequence of (2) we note that 9 is the only prime multiple of 9 which is the sum of two cubes >0; from (3) the only solution $x>0$, $y>0$ of $x^3+y^3=p^2$, p prime, is $(x, y, p)=(1, 2, 3)$, etc. It is not difficult to obtain from (1)—(3) the known types of impossible equations $x^3\pm y^3=n$, except when n is a cube, and some others that do not seem to have been stated.

THE UNIVERSITY OF WASHINGTON

CONTACT CURVES OF THE RATIONAL PLANE CUBIC*

BY L. W. GRIFFITHS

1. *Introduction.* Contact conics and hyperosculating curves of the rational cubic have been discussed by Winger.† Likewise some account has been given of curves of order n which cut the cubic, rational or elliptic, in $(3n-1)$ coincident points.‡ There remains the question of contact curves of order $n\,(n>2)$ whose contacts are of lower orders. This paper considers that question for the rational cubic, with results which hold for $n\geqq 1$ and for contacts of any order.

If the cubic is taken in the canonical form

$$(1) \qquad x_1=3t^2, \qquad x_2=3t, \qquad x_3=t^3+1,$$

a necessary and sufficient condition that a set of $3n$

* Presented to the Society, San Francisco Section, December 22, 1923.

† *Involutions on the rational cubic*, this BULLETIN, vol. 25 (1918), p. 27.

‡ Winger, *Some generalizations of the satellite theory*, this BULLETIN, vol. 26 (1919), p. 75.

points on the cubic be the intersection of a curve* of order n and the cubic is

$$s_{3n} = (-1)^n, \tag{2}$$

where s_{3n} is the product of the $3n$ parameters of the points. The intersecting curve will be a contact curve if the $3n$ points are not all distinct, i. e., if the exponent of at least one parameter in s_{3n} is >1. For instance, hyperosculating curves correspond to the coincidence case $t_1 = t_2 = \cdots = t_{3n}$. It is from (2) that the results in this paper are derived.

2. *An Example.* As an example, consider a contact quartic and require that the 12 points common to it and the cubic coincide in two contacts, a 5-point contact at a point whose parameter is τ and a 7-point contact at a point whose parameter is t. Then, in virtue of (2), t and τ satisfy the equation

$$\tau^5 t^7 = (-1)^{12}. \tag{3}$$

Thus for a given τ there are in fact seven points t, which means that seven quartics can have 5-point contact at an arbitrary point of the cubic and 7-point contact elsewhere. However, it is easily seen from (3) that τ is merely one of a set of 5 points each of which determines this same set of 7 points t_i. That is, here are 35 related contact quartics, 5 with 7-point contact at each t_i and 7 with 5-point contact at each τ_i. It is easily verified also that the 12 parameters of the 12 points τ_i and t_i (in terms of τ) satisfy (2). That is, the 12 contacts of the 35 related contact quartics lie on a non-contact quartic. But τ was an arbitrary point of the cubic. Therefore all contact quartics of the type illustrated are related in sets of 35 each, and the 12 contacts for any one set lie on a quartic.

Similar relations hold for other types of contact quartics. For instance, if τ be supposed a 4-point contact and t an 8-point contact, it can be shown that all contact quartics

* The curve will in fact be one of a pencil when $n>2$. Hereafter the word curve is used with this understanding.

of this type are related in sets of 32 each, etc. Indeed, similar relations hold if and only if the 12 points common to the quartic and the cubic coincide in two contacts.

3. *Generalization.* These relations on contact quartics may be generalized immediately, for the method of proof is independent of the order n of the intersecting curve. Here it is convenient to give the name *complementary contacts* to a pair of points like τ and t which absorb all the intersections of the curve and the cubic; and to indicate that a definite number $r < 3n$ of single intersections coincide at τ by saying that τ is a P^r for the curve in question. Then two complementary contact curves are said to be of the *same type* if their values of n and r are respectively equal.

THEOREM I. *All complementary contact curves of the same type are related in sets of r $(3n-r)$ each, where r and n have the values for the type in question. The distinct contacts for any one such set of complementary contact curves are $3n$ in number, r of them being contacts P^r and the remaining $(3n-r)$ being contacts P^{3n-r}. Each point of the type P^r is a contact P^r for $(3n-r)$ of the curves of the set and for no other complementary contact curves of this type in this set or any other; similarly each point of the type P^{3n-r} is a contact P^{3n-r} for r of the curves of the set. That is, points of the cubic are related in non-overlapping sets of $3n$ each by complementary contact curves of a given type.*

THEOREM II. *The $3n$ contacts of a set of related complementary contact curves lie on a non-contact curve of order n when and only when n is even.*

4. *Several Sets of Contacts.* In contrast to this relation (Theorem II) among the $3n$ contacts of a single set of related complementary contact curves, there are relations between several such sets of contacts: for instance, in the satellite theory. In particular, the theorem that the tangentials of 3 points of a line are collinear relates 3 sets of contacts: each set consists of 3 points, a P^1 (the single

intersection of a tangent and the cubic) and two P^2's (the contacts of tangents to the cubic from the P^1). It will be clear from Theorem 3 that the complete relation between these 3 sets of contacts is rather this theorem on tangentials *and* the less familiar theorem that the 6 contacts of tangents drawn from 3 collinear points lie by threes on 4 lines.

For the general case the several sets of contacts are determined as follows. First choose arbitrary integer values for n, $r<3n$, k. Let τ_i $(i=1, 2, 3, \cdots, 3k)$ be the $3k$ distinct points in which a non-contact curve of order k intersects the cubic. Then (Theorem I) determine the $3n$ distinct contacts of the set of related complementary contact curves of order n in which τ_1 is a P^r; similarly determine the set in which τ_2 is a P^r, etc. Since the τ_i are distinct, there are $3k$ distinct sets of complementary contact curves of the same type, each set of complementary contact curves having its set of $3n$ contacts. These are the contacts whose relations are required. Note that they are 3^2kn distinct points, of which $3kr$ are contacts of the type P^r and $3k(3n-r)$ are contacts of the type P^{3n-r}.

THEOREM III. *If, as above, any $3k$ contacts of the type P^r lie on a curve of order k, then the totality of contacts of the type P^r lie on precisely r^{3k-1} curves of order k; and, further, the totality of contacts of the type P^{3n-r} lie on precisely $(3n-r)^{3k-1}$ curves of order k.*

The proof consists in exhibiting two auxiliary sets of points, called the points T_i and T_i' $(i=1, 2, \cdots, 3k)$, such that (1) a necessary condition that the points τ_i lie on a curve of order k is that the points T_i lie on a curve of order k; (2) a sufficient condition that another selection of points of the type P^r lie on a curve of order k is that the T_i lie on a curve of order k; (3) a sufficient condition that the points T_i' lie on a curve of order k is that the points T_i lie on a curve of order k; and (4) a sufficient condition that a selection of $3k$ points of the type P^{3n-r} lie on a curve of order k is that the points T_i' lie on a curve of order k. Define $T_i = -(-\tau_i)^r$. Then if S_{3k} re-

presents the product of the parameters of the points T_i

$$S_{3k} = (-1)^k(-1)^{kr}(\tau_1\,\tau_2\cdots\tau_{3k})^r. \tag{4}$$

And so $S_{3k} = (-1)^k$ if $(\tau_1\,\tau_2\cdots\tau_{3k}) = (-1)^k$. This proves (1). The proof of (2) is more difficult, and requires the specific parameters of the $3n$ contacts determined by τ_i. By a method similar to that used in discussing quartics they are found to be

$$[\tau_i, \omega_r\tau_i, \omega_r^2\tau_i, \cdots, \omega_r^{r-1}\tau_i, t_i, \omega_{3n-r}t_i, \omega_{3n-r}^2t_i, \cdots, \omega_{3n-r}^{3n-r-1}t_i], \tag{5}$$

where ω_r is a primitive rth root of unity, ω_{3n-r} a primitive $(3n-r)$th root of unity, and $t_i = [(-1)^n/\tau_i^r]^{1/(3n-r)}$, also a primitive root. Now select $(3k-1)$ points, one from among $(\tau_1, \omega_r\tau_1, \cdots, \omega_r^{r-1}\tau_1)$, one from among $(\tau_2, \omega_r\tau_2, \cdots, \omega_r^{r-1}\tau_2)$, etc., omitting one from the set to which τ_j belongs. By these points a curve of order k is determined. It will be shown that the $3k$th point A in which this curve intersects the cubic is some one of the points $(\tau_j, \omega_r\tau_j, \cdots, \omega_r^{r-1}\tau_j)$. For, determine the points T for the points in which this curve intersects the cubic. They are $T_1, \cdots, T_{j-1}, T_a, T_{j+1}, \cdots, T_{3k}$. But by the hypothesis and proof of (1) the $3k$ points $T_1, \cdots, T_{j-1}, T_j, T_{j+1}, \cdots, T_{3k}$ lie on a curve of order k. Since $(3k-1)$ points on the cubic determine a curve of order k, T_a is identical with T_j. Now from the definition of T_j it is clear that each of the r points $(\tau_j, \omega_r\tau_j, \cdots, \omega_r^{r-1}\tau_j)$ determines this same T_j, and that no other points determine this T_j. Therefore A is some one of these points, say $\omega_r^m\tau_j$. But A and the $(3k-1)$ points selected lie on a curve of order k. Therefore (2) is proved. This method of proof of (2) leads to the enumeration r^{3k-1} stated in the theorem. For the choice from the points $(\tau_i, \omega_r\tau_i, \cdots, \omega_r^{r-1}\tau_i)$ can be made independently in r ways and independently of choosing from any of the other sets.

To prove (3) define $T_i' = -(-t_i)^{3n-r}$, where t_i has the value in (5). Then each of the $(3n-r)$ points $(t_i, \omega_{3n-r}t_i, \cdots, \omega_{3n-r}^{3n-r-1}t_i)$ gives the same point T_i', and they are the only ones giving this T_i'. Then if S_{3k}' re-

presents the product of the parameters of the points T_i', it can be shown that $S_{3k}' = 1/S_{3k}$. Therefore (3) is proved. The proof of (4) is precisely the proof of (2), where t_i and T_i' replace τ_i and T_i respectively. The enumeration $(3n-r)^{3k-1}$ is also immediate. The proof of Theorem 3 is therefore complete.

It should be noted that some of the results of Theorem 3, but no enumerations, follow from the theory of residuation.

5. *Several Contacts of the Same Order.* Finally, consider contact curves whose intersections are not completely absorbed in two contacts. Then at least two contacts are arbitrary, and relations are correspondingly indeterminate. The interesting case of precisely three contacts is that in which the contacts are of the same order; then from (2)

$$(t_1 t_2 t_3)^n = (-1)^n. \tag{6}$$

Thus the necessary and sufficient condition that a triad of points be contacts of this sort is that it belong to one of the n involutions

$$s_3 = -\omega_n^i, \quad (i = 0, 1, \cdots, n-1), \tag{7}$$

where ω_n is a primitive nth root of unity. For $n = 2$ these contact curves are the familiar tritangent conics.

The interesting case of n contacts is also that in which the contacts are of the same order, when from (2)

$$(t_1 t_2 \cdots t_n)^3 = (-1)^n, \tag{8}$$

$$s_n = (-1)^n \omega_3^i, \qquad (i = 0, 1, 2), \tag{9}$$

where ω_3 is a primitive cube root of unity.

6. *Analogous Theorems for the General Cubic.* For the general cubic, expressed rationally in terms of $\wp(u)$, analogous theorems hold. The equations in terms of the elliptic argument are not of the simple type (2) afforded by the rational cubic, but yield determinate enumerations.

THE UNIVERSITY OF WASHINGTON

NOTE ON THE PROJECTIVE GEOMETRY OF PATHS

BY T. Y. THOMAS*

1. *Projective Geometry of Paths.* It was first shown by Weyl† that the functions $\Gamma^i_{\alpha\beta}$ and the functions

$$(1) \qquad \Lambda^i_{\alpha\beta} = \Gamma^i_{\alpha\beta} + \delta^i_\alpha \psi_\beta + \delta^i_\beta \psi_\alpha,$$

where ψ_α is an arbitrary covariant vector, and

$$\delta^i_\alpha = 0, \quad \text{for } i \neq \alpha; \qquad = 1, \quad \text{for } i = \alpha,$$

define the same geometry of paths. This leads to the consideration of properties of the paths which are independent of the particular set of functions $\Gamma^i_{\alpha\beta}$ by means of which the paths are defined. Theorems expressing such properties constitute the projective geometry of paths. In the following note we give a few theorems belonging to the projective geometry of paths.

2. *Projective Tensors.* Theorems of the projective geometry of paths appear to have their statement in terms of what may be called *projective tensors*, i. e. tensors which are independent of the particular set of functions $\Gamma^i_{\alpha\beta}$ defining the paths. We shall show how a set of projective tensors may be derived by covariant differentiation from an n-uple of mutually independent vectors.

Let $h_{(\alpha)i}$ denote an n-uple of independent covariant vectors. Then the determinant

$$(2) \qquad h = |h_{(\alpha)i}|$$

does not vanish identically. We may therefore define an n-uple of contravariant vectors $h^{(\alpha)i}$ as the cofactors of the

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† H. Weyl, Göttinger Nachrichten, 1921, p. 99. See also O. Veblen, Proceedings of the National Academy, vol. 8 (1922), p. 347; and O. Veblen and T. Y. Thomas, Transactions of this Society, vol. 25, p. 557.

corresponding $h_{(\alpha)i}$ in the determinant h divided by h. Hence

$$h_{(\alpha)i}\, h^{(\alpha)j} = \delta_i^j, \tag{3}$$

and

$$h_{(i)\alpha}\, h^{(j)\alpha} = \delta_i^j, \tag{4}$$

where the left members represent a summation in α from $\alpha = 1$ to $\alpha = n$. It will be understood in the following that each index which appears twice in a term, once as a subscript and once as a superscript, is to be summed over the values 1 to n.

The change $[h_{(\alpha)i,j}]$ in the covariant derivative $h_{(\alpha)i,j}$ of the n-uple $h_{(\alpha)i}$, based on the functions $\Gamma^i_{\alpha\beta}$, when the Γ's are replaced by the above functions A, is

$$[h_{(\alpha)i.j}] = -h_{(\alpha)i}\,\psi_j - h_{(\alpha)j}\,\psi_i. \tag{5}$$

Hence

$$(n+1)\,\psi_i = -h^{(\alpha)\beta}\,[h_{(\alpha)\beta,\,i}]. \tag{6}$$

The vector γ_i defined by

$$(n+1)\gamma_i = -h^{(\alpha)\beta}\,h_{(\alpha)\beta,\,i} \tag{7}$$

will therefore change by ψ_i when $\Gamma^i_{\alpha\beta}$ is replaced by $A^i_{\alpha\beta}$, i.e.

$$[\gamma_i] = \psi_i. \tag{8}$$

Hence

$$h_{(\alpha)ij} = h_{(\alpha)i,j} + h_{(\alpha)i}\gamma_j + h_{(\alpha)j}\gamma_i \tag{9}$$

represents a set of n projective covariant tensors of the second order.

In a similar way we may treat the case of an n-uple of independent contravariant vectors $h^{(\alpha)i}$. This leads to a set of n projective mixed tensors of the second order given by

$$h^{(\alpha)i}_j = h^{(\alpha)i}{}_{,j} - h^{(\alpha)i}\lambda_j - \delta^i_j h^{(\alpha)\beta}\lambda_\beta \tag{10}$$

where the vector λ_i is defined as

$$(n+1)\lambda_i = h_{(\alpha)\beta}h^{(\alpha)\beta}{}_{,\,i}, \tag{11}$$

in which the covariant n-uple $h_{(\alpha)i}$ is obtained from the

contravariant n-uple $h^{(\alpha)i}$ by dividing the cofactors of the elements of the determinant

$$h = |h^{(\alpha)i}|$$

by the determinant h. Other projective tensors will appear in the following paragraphs.

3. *The n-uple of Parallel Vectors.** If $h_{(\alpha)i}$ represents an n-uple of independent parallel covariant vectors for some set of functions $\Gamma^i_{\alpha\beta}$ defining the paths, then

$$h_{(\alpha)i|j} = 0 \tag{12}$$

identically. The condition (12) is also sufficient for the n-uple of independent covariant vectors $h_{(\alpha)i}$ to be parallel. Hence we have the following theorem.

THEOREM I. *A necessary and sufficient condition for the n-uple of independent covariant vectors $h_{(\alpha)i}$ to be parallel in the projective geometry of paths is that* (12) *be satisfied.*

Forming the equations

$$h^{(\alpha)i}_{|j} = 0, \tag{13}$$

we may state the corresponding theorem for the case of a contravariant n-uple.

THEOREM II. *A necessary and sufficient condition for the n-uple of independent contravariant vectors $h^{(\alpha)i}$ to be parallel in the projective geometry of paths is that* (13) *be satisfied.*

4. *Reduction to the Euclidean Geometry.* The affine geometry of paths becomes a euclidean geometry if there exists an n-uple of independent parallel covariant vectors $h_{(\alpha)i}$. This is obviously a necessary condition and it is seen to be sufficient since the existence of an n-uple of parallel vectors $h_{(\alpha)i}$ leads to the equations

$$h_{(i)\sigma} B^{\sigma}_{\alpha\beta\gamma} = 0, \tag{14}$$

where $B^{\sigma}_{\alpha\beta\gamma}$ is the affine (ordinary) curvature tensor, and hence

* L. P. Eisenhart, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), p. 207. See also O. Veblen and T. Y. Thomas, loc. cit., p. 589.

$$B^i_{\alpha\beta\gamma} = 0, \tag{15}$$

owing to the independence of the vectors $h_{(\alpha)i}$. Hence we have the following theorem.

THEOREM III. *A necessary and sufficient condition for the projective geometry of paths to be a euclidean geometry is that the Γ's be such that there exists an n-uple of independent covariant vectors $h_{(\alpha)i}$ satisfying* (12).

A similar theorem may be stated for the n-uple of contravariant vectors.

5. *Reduction to the Riemann Geometry.* For the projective geometry of paths to be a Riemann geometry it is necessary and sufficient that the Γ's be such that there exists a symmetric tensor $g_{\alpha\beta}$ whose determinant

$$g = |g_{\alpha\beta}| \tag{16}$$

does not vanish identically, and a vector ψ_i such that

$$\tilde{g}_{\alpha\beta,\gamma} = 0, \tag{17}$$

where $\tilde{g}_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $A^i_{\alpha\beta}$ given by (1). Writing (17) in the form

$$g_{\alpha\beta,\gamma} - 2g_{\alpha\beta}\psi_\gamma - g_{\gamma\beta}\psi_\alpha - g_{\alpha\gamma}\psi_\beta = 0, \tag{18}$$

where $g_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $\Gamma^i_{\alpha\beta}$, and multiplying by the contravariant tensor $g^{\alpha\beta}$, formed in the ordinary manner from the tensor $g_{\alpha\beta}$, we have

$$2(n+1)\psi_\gamma = g^{\alpha\beta}g_{\alpha\beta,\gamma}. \tag{19}$$

Substituting this value of ψ_γ in the left member of (18) and denoting the resulting expression by $g_{\alpha\beta\gamma}$, we find

$$g_{\alpha\beta\gamma} = g_{\alpha\beta,\gamma} - \frac{g^{\mu\nu}}{n+1}\left(g_{\alpha\beta}g_{\mu\nu,\gamma} + \frac{1}{2}g_{\gamma\beta}g_{\mu\nu,\alpha} + \frac{1}{2}g_{\alpha\gamma}g_{\mu\nu,\beta}\right). \tag{20}$$

Hence (18) takes the form

$$g_{\alpha\beta\gamma} = 0. \tag{21}$$

The tensor $g_{\alpha\beta\gamma}$ defined by (20) may be shown to be projective.

The equations (21) constitute a necessary condition for the projective geometry of paths to be a Riemann geometry based on the tensor $g_{\alpha\beta}$ as the fundamental metric tensor. This condition is easily shown to be sufficient. Hence we have the following theorem.

THEOREM IV. *A necessary and sufficient condition for the projective geometry of paths to be a Riemann geometry is that the Γ's be such that there exists a tensor $g_{\alpha\beta}$ which satisfies the equations* (21).

6. *Reduction to the Weyl Geometry.** Let us denote by $g_{\alpha\beta}$ a covariant symmetric tensor whose determinant g does not vanish identically, as in the preceding paragraph, and also by φ_α a covariant vector. Let us then form the equations

$$g_{\alpha\beta\gamma} = g_{\alpha\beta}\varphi_\gamma, \tag{22}$$

where $g_{\alpha\beta\gamma}$ is now defined by

$$g_{\alpha\beta\gamma} = g_{\alpha\beta,\gamma} - \frac{g^{\mu\nu}}{n+1}\left(g_{\alpha\beta}g_{\mu\nu,\gamma} + \frac{1}{2}g_{\gamma\beta}g_{\mu\nu,\alpha} + \frac{1}{2}g_{\alpha\gamma}g_{\mu\nu,\beta}\right) + \frac{n}{n+1}\left(g_{\alpha\beta}\varphi_\gamma + \frac{1}{2}g_{\gamma\beta}\varphi_\alpha + \frac{1}{2}g_{\alpha\gamma}\varphi_\beta\right), \tag{23}$$

in which $g_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $\Gamma^i_{\alpha\beta}$. The tensor $g_{\alpha\beta\gamma}$ defined by (23) is a projective tensor. We may prove the following theorem.

THEOREM V. *A necessary and sufficient condition for the projective geometry of paths to be a Weyl geometry is that the Γ's be such that there exists a tensor $g_{\alpha\beta}$ and vector φ_α which satisfy the equations* (22).

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* By the Weyl geometry is meant the geometry used by Weyl as the basis of his combined theory of gravitation and electricity. See H. Weyl, *Raum, Zeit, Materie*, 4th ed., p. 113.

THE TENSOR CHARACTER OF THE GENERALIZED KRONECKER SYMBOL*

BY F. D. MURNAGHAN

1. *Introduction.* In a previous paper† we have considered the use of the generalized Kronecker symbol $\delta^{r_1 r_2 \cdots r_m}_{s_1 s_2 \cdots s_m}$ in presenting the theory of determinants and we now proceed to show that it is an arithmetic tensor of the type indicated by its subscripts and superscripts, i. e., it is covariant of rank m and contravariant of rank m. By the statement that a tensor is arithmetic, we mean that its presentation is independent of the particular coordinate system in use and that it has the same numerical values for its various components at all points of space.‡

The generalized Kronecker symbol may be defined by means of the equation

$$(1) \qquad \delta^{r_1 r_2 \cdots r_m}_{s_1 s_2 \cdots s_m} = \begin{vmatrix} \delta^{r_1}_{s_1} & \cdots & \delta^{r_1}_{s_m} \\ \delta^{r_2}_{s_1} & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \delta^{r_m}_{s_1} & \cdots & \delta^{r_m}_{s_m} \end{vmatrix}.$$

Here the labels r and s run independently over a set of n numbers $1, 2, \cdots, n$ and $\delta^r_s = 1$ if $r = s$, and $= 0$ if $r \neq s$; δ^r_s is the ordinary Kronecker symbol and it is usually denoted by g^r_s in the theory of relativity. It is there derived as the scalar product of the metric tensor g_{rs} and its reciprocal g^{rs}, but this mode of presentation is somewhat

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† *The Generalized Kronecker symbol and its application to the theory of determinants*, AMERICAN MATHEMATICAL MONTHLY vol. 32 (1925), p. 233. This paper will be denoted by the symbol (*A*) in references below.

‡ Cf. P. FRANKLIN, PHILOSOPHICAL MAGAZINE, (6), vol. 45 (1923), p. 998.

unfortunate since the tensor character of the symbol has nothing to do with the metric properties of the space.

The simplest procedure in proving the tensor character of the symbol $\delta^{r_1 r_2 \cdots r_m}_{s_1 s_2 \cdots s_m}$ is first to prove the theorem when $m = n$, the number of dimensions of the space. To do this we shall define a tensor of rank $2n$, contravariant of rank n and covariant of rank n, by the statement that its presentation in a particular coordinate system $(x^1, x^2, x^3, \cdots, x^n)$ is furnished by the values of the generalized Kronecker symbol $\delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$. Denoting the presentation of this tensor in any other coordinate system $(y^1, y^2, \cdots, y^n)$ by $\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$, we have to show that

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$$

where the labels $(r_1, r_2, \cdots, r_n)$ and $(s_1, s_2, \cdots, s_n)$ may each be assigned independently any one of the values $(1, 2, \cdots, n)$. We have, from the definition of a tensor,

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}}$$

where we have adopted the convention of the previous paper (*A*), according to which *Greek* labels occurring twice in any term indicate summations. Writing*

$$\delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_n} = \frac{1}{n!} \delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\lambda_1 \lambda_2 \cdots \lambda_n} \delta^{\lambda_1 \lambda_2 \cdots \lambda_n}_{\beta_1 \beta_2 \cdots \beta_n} \tag{2}$$

we have

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \frac{1}{n!} \left(\delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\lambda_1 \lambda_2 \cdots \lambda_n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}} \right) \left(\delta^{\lambda_1 \lambda_2 \cdots \lambda_n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}} \right).$$

Now the expression on the right hand side is a summation of products of two factors of the type

$$\delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{l_1 l_2 \cdots l_n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}} \quad \text{and} \quad \delta^{l_1 l_2 \cdots l_n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}}$$

* See paper (*A*), equation (2·3).

respectively. Each of these factors is a determinant of order n (see (3·4), paper (*A*)) and since an inversion of any two of the labels $(l_1, l_2, \cdots, l_n)$ merely changes the sign of both factors we may write

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{1\,2\,\cdots\,n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}} \cdot \delta^{1\,2\,\cdots\,n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}}.$$

The product of the two determinants on the right is a determinant of which the element in the pth row and the qth column is

$$\frac{\partial y^{r_p}}{\partial x^{\sigma}} \cdot \frac{\partial x^{\sigma}}{\partial y^{s_q}} \quad \text{or} \quad \delta^{r_p}_{s_q}. \tag{3}$$

Hence

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$$

which proves the theorem stated.

The tensor character of the symbols for the cases $m < n$ follows at once by contraction of the tensor $\delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$. Thus it is immediately apparent that

$$\delta^{r_1 r_2 \cdots r_{n-1}}_{s_1 s_2 \cdots s_{n-1}} = \delta^{r_1 r_2 \cdots r_{n-1} \alpha}_{s_1 s_2 \cdots s_{n-1} \alpha} \tag{4}$$

for in the summation on the right all the terms vanish unless $(r_1, r_2, \cdots, r_{n-1})$ are all different and $(s_1, s_2, \cdots, s_{n-1})$ all different and also α different from any of the $(r_1, r_2, \cdots, r_{n-1})$ and the $(s_1, s_2, \cdots, s_{n-1})$. The only term which has a value different from zero occurs, therefore, when $(r_1, r_2, \cdots, r_{n-1})$ and $(s_1, s_2, \cdots, s_{n-1})$ are arrangements of the same group of $n-1$ out of the n numbers $(1, 2, \cdots, n)$ and α is the remaining number. The equation (4) shows that $\delta^{r_1 r_2 \cdots r_{n-1}}_{s_1 s_2 \cdots s_{n-1}}$ is an arithmetic mixed tensor contravariant of rank $n-1$ and covariant of rank $n-1$. Proceeding similarly we arrive at the simple Kronecker symbol $\delta^r_s = \delta^{r \alpha_1 \alpha_2 \cdots \alpha_{n-1}}_{s \alpha_1 \alpha_2 \cdots \alpha_{n-1}}$ which is an arithmetic mixed tensor of the second rank.

It may not be superfluous to call attention again to the fact that these tensors are non-metric. The space for which

they are defined is the general space of analysis situs, or topology, in which a point is merely a set of n ordered numbers. The usual presentation of, and notation for, the mixed tensor δ^r_s is therefore unfortunate. This starts with a symmetric covariant tensor g_{rs} of the second rank which furnishes the metric ground form $(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ of a Riemann space. From this is derived a contravariant presentation g^{rs} of the metric tensor and the inner product $g^r_s = g^{r\alpha}g_{\alpha s}$ furnishes the mixed arithmetic tensor which we have denoted by δ^r_s.

2. *The Outer Multiplication of Tensors.* If we have two covariant tensors $a_{r_1 r_2 \cdots r_p}$ and $b_{s_1 s_2 \cdots s_q}$ of ranks p and q respectively we may derive from them, by means of the generalized Kronecker tensor, an alternating covariant tensor of rank $p+q$ as follows:

$$(5) \qquad c_{r_1 r_2 \cdots r_p s_1 \cdots s_q} = \delta^{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q}_{r_1 \cdots r_p s_1 \cdots s_q} a_{\alpha_1 \cdots \alpha_p} b_{\beta_1 \cdots \beta_q};$$

$p+q$ must be $\leqq n$ in order that the alternating tensor thus arrived at may not vanish identically.

Similarly we derive from two contravariant tensors $a^{r_1 \cdots r_p}$ and $b^{s_1 \cdots s_q}$ of ranks p and q respectively an alternating contravariant tensor of rank $p+q$

$$(6) \qquad c^{r_1 \cdots r_p s_1 \cdots s_q} = \delta^{r_1 \cdots r_p s_1 \cdots s_q}_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q} a^{\alpha_1 \cdots \alpha_p} b^{\beta_1 \cdots \beta_q}.$$

If the original tensors are alternating considerable simplification results. It is convenient to remove a numerical factor $p!\,q!$ and we may define the derived tensor $c_{r_1 \cdots r_p s_1 \cdots s_q}$ by the equations

$$(7) \qquad c_{r_1 \cdots r_p s_1 \cdots s_q} = \frac{1}{p!}\frac{1}{q!}\delta^{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q}_{r_1 \cdots r_p s_1 \cdots s_q} a_{\alpha_1 \cdots \alpha_p} b_{\beta_1 \cdots \beta_q},$$

$$= \sum_{l,m} \delta^{l_1 \cdots l_p m_1 \cdots m_q}_{r_1 \cdots r_p s_1 \cdots s_q} a_{l_1 \cdots l_p} b_{m_1 \cdots m_q}.$$

In the last expression $(l_1, l_2, \cdots, l_p)$ is any group of p out of the $p+q$ numbers $(r_1, \cdots, r_p, s_1, \cdots, s_q)$ and $(m_1, \cdots, m_q)$

is the remaining group of q numbers. No group $(l_1, \cdots, l_p)$ is to be repeated in the summation. The tensor derived in this way may be called the outer product of the two alternating covariant tensors. Similarly the outer product of two alternating contravariant tensors is given by

$$(8) \qquad c^{r_1 \cdots r_p s_1 \cdots s_q} = \frac{1}{p!}\frac{1}{q!}\delta^{r_1 \cdots r_p s_1 \cdots s_q}_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q} a^{\alpha_1 \cdots \alpha_p} b^{\beta_1 \cdots \beta_p}$$

$$= \sum_{l, m} \delta^{r_1 \cdots r_p s_1 \cdots s_q}_{l_1 \cdots l_p m_1 \cdots m_q} a^{l_1 \cdots l_p} b^{m_1 \cdots m_q}.$$

The alternating tensor derived as in (5) from two non-alternating tensors is not essentially different from the outer product of two alternating tensors. For we may write it, on using the result*

$$\delta^{l_1 \cdots l_p m_1 \cdots m_q}_{r_1 \cdots r_p s_1 \cdots s_q} = \frac{1}{p!}\frac{1}{q!}\delta^{l_1 \cdots l_p}_{\lambda_1 \cdots \lambda_p}\delta^{m_1 \cdots m_q}_{\mu_1 \cdots \mu_q}\delta^{\lambda_1 \cdots \lambda_p}_{r_1 \cdots r_p}\delta^{\mu_1 \cdots \mu_q}_{s_1 \cdots s_q},$$

in the form

$$\frac{1}{p!}\frac{1}{q!}\delta^{\lambda_1 \cdots \lambda_p \mu_1 \cdots \mu_q}_{r_1 \cdots r_p s_1 \cdots s_q}\left(\delta^{\alpha_1 \cdots \alpha_p}_{\lambda_1 \cdots \lambda_q} a_{\alpha_1 \cdots \alpha_p}\right)\left(\delta^{\beta_1 \cdots \beta_q}_{\mu_1 \cdots \mu_q} b_{\beta_1 \cdots \beta_q}\right)$$

showing that $c_{r_1 \cdots r_p s_1 \cdots s_q}$, as defined in (5), is really the outer product of the two alternating tensors

$$f_{t_1 \cdots t_p} = \delta^{\alpha_1 \cdots \alpha_p}_{t_1 \cdots t_p} a_{\alpha_1 \cdots \alpha_p}$$

and

$$g_{u_1 \cdots u_q} = \delta^{\beta_1 \cdots \beta_q}_{u_1 \cdots u_q} b_{\beta_1 \cdots \beta_q}.$$

It is well known that alternating covariant tensors may be used as the coefficients of invariant integral forms. In fact, we may, from p arbitrary contravariant tensors $a^r, b^s, \cdots, g^t$, of rank one, form the alternating contravariant tensor of rank p

$$e^{r_1 \cdots r_p} = \delta^{r_1 \cdots r_p}_{\alpha_1 \cdots \alpha_p} a^{\alpha_1} b^{\alpha_2} \cdots g^{\alpha_p}.$$

When $(a^r, b^s, \cdots, g^t)$ are differential vectors tangent to the parametric lines of a spread of p dimensions so that

* See equation (2·4bis), paper (*A*).

$$a^r = \frac{\partial x^r}{\partial u_1} du_1,$$

etc., the scalar product $a_{\varrho_1 \cdots \varrho_p} e^{\varrho_1 \cdots \varrho_p}$ is called *an integral form of order* p (in this case $e^{r_1 \cdots r_p}$ is usually denoted by $d(x^{r_1} x^{r_2} \cdots x^{r_p})$). Its integral over the spread is called *the integral of the alternating covariant tensor* $a_{r_1 \cdots r_p}$ *over the spread of* p *dimensions.* In this connection the result (7), which enables us to derive from two integral forms of orders p and q respectively an integral form of order $p+q$, is known as the *law of the outer product of two integral forms.**

3. *Connection between the Kronecker Tensor and Reciprocation with respect to a Quadratic Differential Form.* If we have any symmetric covariant tensor of the second rank a_{rs} we may denote by a the value of the determinant which has a_{rs} as the element in its rth row and sth column. It follows at once that the product $\sqrt{a}\, d(x^1 x^2 \cdots x^n)$ is invariant, and so we may introduce an alternating covariant tensor $\varepsilon_{r_1 \cdots r_n}$ which is defined by the statement that its components have the value $\pm \sqrt{a}$ according as the arrangement $(r_1, r_2, \cdots, r_n)$ of the n numbers $(1, 2, \cdots, n)$ is of the even or odd class. For the integral form

$$\varepsilon_{\alpha_1 \cdots \alpha_n} d(x^{\alpha_1} x^{\alpha_2} \cdots x^{\alpha_n})$$

is invariant, its value being $n! \sqrt{a}\, d(x^1 x^2 \cdots x^n)$. From this alternating covariant tensor we derive an alternating contravariant tensor of rank n as follows. Solving the system of n^2 equations

$$a_{r\alpha} a^{\alpha s} = \delta_r^s$$

for the n^2 unknowns a^{rs}, we obtain a contravariant tensor of rank two which is said to be the reciprocal of a_{rs} with respect to the quadratic differential form $a_{\alpha\beta} dx^\alpha dx^\beta$. The tensor $\varepsilon^{r_1 \cdots r_n} = a^{r_1 \alpha_1} a^{r_2 \alpha_2} \cdots a^{r_n \alpha_n} \varepsilon_{\alpha_1 \cdots \alpha_n}$ is said to be the

* Reference may be made to E. Cartan, *Leçons sur les Invariants Intégraux*, Paris, 1922. E. Cartan, ANNALES DE L'ECOLE NORMALE, 1899. H. Bateman, *Differential Equations*, Chapter 7. London, 1918. H. Bateman, PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 8 (1910).

reciprocal of $\varepsilon_{r_1 \cdots r_n}$ with respect to the quadratic differential form. It is alternating and has the value $\pm 1/\sqrt{a}$ according as the arrangement $(r_1, \cdots, r_n)$ of the n numbers $(1, 2, \cdots, n)$ is of the even or odd class. Then the simple product $\varepsilon^{r_1 \cdots r_n} \varepsilon_{s_1 \cdots s_n}$ is the generalised Kronecker tensor

$$\delta^{r_1 \cdots r_n}_{s_1 \cdots s_n}.$$

It may be remarked that the operation of finding tensors reciprocal to any quadratic differential form is a possible one so long as the differential form is non-singular ($a \neq 0$). When the quadratic differential form is the metrical ground form $(ds)^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ of a Riemann space we may say that the reciprocal covariant and contravariant tensors are but different representations of the same physical idea which is the tensor. This is by analogy with the case of rectangular cartesian coordinates in euclidean space where the distinction between covariance and contravariance breaks down and the components of two tensors which are reciprocal with respect to the metric ground form $(ds)^2 = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$ coincide.

We have endeavored to show in the preceding paragraphs the utility of the alternating tensor $\delta^{r_1 \cdots r_m}_{s_1 \cdots s_m}$ in proving theorems of tensor algebra. It will be readily recognised that there is an intimate connection here with Grassmann's *Ausdehnungslehre*, and we believe, in fact, that a systematic exposition of this theory with the aid of the generalized Kronecker symbol would help to make it more widely understood. The use of the symbol in connection with the discussion of the orientation of cells in analysis situs* is also recommended.

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* O. Veblen. *Cambridge Colloquium Lectures*, New York, 1922.

TWO GENERAL FUNCTIONAL EQUATIONS*

BY W. H. WILSON

The object of this paper is to discuss the functional equations

(1) $f(x+y)+g(x-y)=h(x)k(y), \quad h(x)\not\equiv 0, \quad k(x)\not\equiv 0,$

and

(2) $F(x+y)G(x-y)=H(x)+K(y), \quad F(x)\not\equiv 0, \quad G(x)\not\equiv 0,$

in which x and y are independent variables and $f(x)$, $g(x)$, $h(x)$, $k(x)$, $F(x)$, $G(x)$, $H(x)$, $K(x)$ are functions to be determined. Special cases of equations (1) and (2) have been discussed in the literature. Some of the more familiar special cases are

$$h(x)\equiv k(x)\equiv f(x)\equiv \psi(x), \quad g(x)\equiv 0,$$
$$g(x)\equiv k(x)\equiv f(x), \quad h(x)\equiv 2f(x),$$
$$G(x)\equiv 1, \; H(x)\equiv K(x)\equiv F(x)\equiv \varphi(x),$$
$$G(x)\equiv F(x), \; H(x)\equiv F^2(x), \; K(x)\equiv F^2(x)-1,$$
$$G(x)\equiv F(x), \; H(x)\equiv F^2(x), \; K(x)\equiv -F^2(x).$$

In this paper no relationships are assumed between the functions in equation (1) or the functions in equation (2). Furthermore, no restrictions (such as continuity, differentiability, etc.) are imposed on the functions. The variables, x and y, are not assumed real nor must they necessarily be complex. The author shows that the functions of equations (1) and (2) are expressible in terms of the functions $\varphi(x)$ and $\psi(x)$ which satisfy the Cauchy equations

(3) $$\varphi(x+y)=\varphi(x)+\varphi(y),$$

(4) $$\psi(x+y)=\psi(x)\psi(y),$$

given above as special cases of (1) and (2). The results of the paper are of sufficient generality to permit immediate

* Presented to the Society, November 29, 1924.

application of information concerning solutions of (3) and (4) to equations (1) and (2).

If equation (1) *is satisfied, there are functions* $F(x)$, $G(x)$, $H(x)$, $K(x)$, *related to the functions* $f(x)$, $g(x)$, $h(x)$, $k(x)$, *which satisfy equation* (2).

To prove this replace x by $x+y$ and y by $x-y$ in equation (1). Then

$$h(x+y)k(x-y) = f(2x)+g(2y)$$

which is equation (2) if we define the functions of (2) by the equations

$$F(x) \equiv h(x), \quad G(x) \equiv k(x), \quad H(x) \equiv f(2x), \quad K(x) \equiv g(2x).$$

In like manner, *if equation* (2) *is satisfied, there are functions* $f(x)$, $g(x)$, $h(x)$, $k(x)$, *related to the functions* $F(x)$, $G(x)$, $H(x)$, $K(x)$, *which satisfy equation* (1). In view of these statements, it is sufficient if we discuss equation (1).

Since $k(x) \not\equiv 0$, there is some a such that $k(a) \neq 0$. We may replace y by $y+a$ in equation (1) with the result

$$f(x+y+a)+g(x-y-a) = h(x)k(y+a). \tag{5}$$

If we write

$$k(a)\bar{f}(x) = f(x+a),$$
$$k(a)\bar{g}(x) = g(x-a),$$
$$k(a)\bar{k}(x) = k(x+a),$$

equation (5) becomes at once of form (1) with the added condition that $\bar{k}(0) = 1$. *We may therefore suppose, in what follows, that this transformation has been made and that* $k(0) = 1$.

In equation (1), let $y = x$. Then

$$f(2x) = h(x)k(x)-g(0). \tag{6}$$

In like manner, if we let $y = -x$ in (1),

$$g(2x) = h(x)k(-x)-f(0). \tag{7}$$

Equations (6) and (7) show that $f(x)$ and $g(x)$ are determined, except for constants, by $h(x)$ and $k(x)$. In (1),

replace x by $x+y$ and y by $x-y$. It follows that

$$h(x+y)k(x-y) = f(2x)+g(2y) = h(x)k(x)+h(y)k(-y)-[f(0)+g(0)]. \tag{8}$$

The auxiliary equation

$$h(2x) = [k(x)+k(-x)]h(x)-[f(0)+g(0)] \tag{9}$$

may be obtained by letting $y = x$ in (8). Interchanging x and y in (8), we have

$$h(x+y)k(y-x) = h(y)k(y)+h(x)k(-x)-[f(0)+g(0)]. \tag{10}$$

If, now, we add equations (8) and (10) and use equation (9), we find

$$h(x+y)[k(x-y)+k(y-x)] = h(2x)+h(2y), \tag{11}$$

which is transformed into

$$h(x+y)+h(x-y) = 2h(x)E(y), \tag{12}$$

where $E(y)$ is the even component of $k(y)$, by replacing $2x$ by $x+y$ and $2y$ by $x-y$. The relation of equation (12) to equations (3) and (4) has been discussed by the writer.*

In order to find the odd component $S(x)$ of $k(x)$, we may subtract equation (10) from equation (8), whence

$$h(x+y)S(x-y) = h(x)S(x)-h(y)S(y). \tag{13}$$

In equation (13), replace x by $-x$ and y by $-y$ and add the equation thus found to (13). If we denote the odd component of $h(x)$ by $S_h(x)$, we have

$$S_h(x+y)S(x-y) = S_h(x)S(x)-S_h(y)S(y). \tag{14}$$

Replacing y by $-y$ in (14) leaves the right-hand member of the equation unchanged. Therefore

$$S_h(x+y)S(x-y) \equiv S_h(x-y)S(x+y), \tag{15}$$

whence, if $S_h(x) \not\equiv 0$,

$$S(x) \equiv \lambda S_h(x), \tag{16}$$

where λ is a constant.

* This BULLETIN, vol. 27 (1920), p. 302, equation (7).

If $S_h(x) \equiv 0$, $h(x)$ is an even function. Replacing x by $-x$ in (1), we have

$$f(-x+y)+g(-x-y) = h(x)k(y) = f(x+y)+g(x-y),$$

from which

$$g(x) = f(-x)+\mu,$$

where μ is a constant. Equation (1) may now be written in the form

$$f(y+x)+f(y-x)+\mu = h(x)k(y). \tag{17}$$

Since $h(x) \not\equiv 0$, there is some value, say b, of x such that $h(b) \neq 0$. Replace x by b and y by $y+x$ in equation (17). The result is

$$h(b)k(y+x) = f(y+x+b)+f(y+x-b)+\mu. \tag{18}$$

If we replace x by $-x$ in (18), we have

$$h(b)k(y-x) = f(y-x+b)+f(y-x-b)+\mu \tag{19}$$

whence, from (18) and (19),

$$\begin{aligned} h(b)[k(y+x)+k(y-x)] &= [f(y+x+b) \\ &\quad +f(y-x-b)+\mu]+[f(y+x-b)+f(y-x+b)+\mu] \\ &= h(x+b)k(y)+h(x-b)k(y) \\ &= [h(x+b)+h(x-b)]k(y). \end{aligned} \tag{20}$$

We may now define a function $C(x)$ by the equation

$$2h(b)C(x) = h(x+b)+h(x-b).$$

It is easily seen that $C(x)$ is an even function. Equation (20) may now be written in the form

$$k(y+x)+k(y-x) = 2C(x)k(y). \tag{21}$$

If we interchange x and y in (21), we have

$$k(x+y)+k(x-y) = 2k(x)C(y), \tag{22}$$

which is of the same form as equation (12). Since we are concerned, however, with the odd component of $k(x)$, we may change x to $-x$ and y to $-y$ in (22), and subtract the equation thus found from (22); whence we shall have

$$S(x+y)+S(x-y) = 2S(x)C(y), \tag{23}$$

which is also of form (12), but which has the advantage of oddness (see reference under (12)).

The relationships of functions $f(x)$, $g(x)$, $h(x)$, $k(x)$ to the functions $\Phi(x)$ and $\psi(x)$ may now be summarized as follows (bearing in mind the transformation by which $k(0) = 1$):

CASE A: $S_h(x) \not\equiv 0$. Either

$$E(x) = \frac{\psi(x)+\psi(-x)}{2}, \qquad S_h(x) = \frac{\psi(x)-\psi(-x)}{2\alpha},$$

where α is a constant different from zero; or else

$$E(x) \equiv 1, \qquad S_h(x) = \Phi(x).$$

The even component of $h(x)$ is $\eta E(x)$, where η is a constant. In addition, we have

$$S(x) = \lambda S_h(x),$$

so that

$$h(x) = \eta E(x) + S_h(x), \qquad \bar{k}(x) = E(x) + S(x),$$
$$k(x) = k(a)\bar{k}(x-a), \qquad k(a) \neq 0,$$
$$f(2x) = h(x)k(x) - g(0), \qquad g(2x) = h(x)k(-x) - f(0).$$

CASE B: $S_h(x) \equiv 0$. In this case we have

$$E(x) = \frac{\psi_1(x)+\psi_1(-x)}{2},$$

and either

$$S(x) = \frac{\psi_2(x) - C(x)}{\alpha},$$

where α is a constant different from zero; or else

$$S(x) = \Phi(x) \text{ and } C(x) \equiv 1,$$

where each of the functions $\psi_1(x)$ and $\psi_2(x)$ satisfies (4) and

$$2h(b)C(x) = h(x+b) + h(x-b), \quad h(b) \neq 0.$$

In this case,

$$h(x) = \eta E(x),$$

where η is a constant; and

$$\bar{k}(x) = E(x) + S(x), \qquad k(x) = \bar{k}(a)k(x-a), \qquad k(a) \neq 0,$$
$$f(2x) = h(x)k(x) - g(0), \qquad g(2x) = h(x)k(-x) - f(0).$$

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FUNCTIONAL INVARIANTS, WITH A CONTINUITY OF ORDER p, OF ONE-PARAMETER FREDHOLM AND VOLTERRA TRANSFORMATION GROUPS*

BY A. D. MICHAL

1. *Object of the Paper.* In the present paper sufficient conditions are given for the invariance of functionals $f[y(\tau_0'), y'(\tau_0'), y''(\tau_0'), \cdots, y^{(p)}(\tau_0')]$ with a Volterra variation under given arbitrary Fredholm groups

$$\delta y(x) = \left[\int_0^1 H(x,s)y(s)\,ds\right]\delta a \tag{1}$$

with a kernel $H(x,s)$ of the type†

$$H(x,s) = \eta(x)\psi(s). \tag{2}$$

These sufficient conditions are given in the form of func-

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† In this BULLETIN, vol. 30 (July, 1924), pp. 338–344, the writer considered analytic functionals $f[y(\tau_0'), y'(\tau_0')]$ admitting Fredholm groups (1). In the process of finding sufficient conditions for invariance, a certain amount of specialization on the kernel $H(x,s)$ was necessary. The specialization however was erroneously made more than necessary due to an error in formula (16). The reader will verify the statement that the corrected formula (16) becomes

$$f_{i,k}(t_1,\cdots,t_i;t_{i+1},\cdots,t_{i+k-1},t)$$
$$=(-1)^k\varphi(t)\varphi(t_{i+k-1})\varphi(t_{i+k-2})\cdots\varphi(t_{i+1})f_{i+k,0}(t_{i+1},t_{i+2},\cdots,t_{i+k-1},t,t_1,\cdots,t_i).$$

With the aid of this corrected formula, we can show readily that the kernel $H(x,s)$ will have to be of the form $\eta(x)\psi(s)$ and hence $\varphi(t) = \eta(t)\Big/\dfrac{d\eta(t)}{dt}$. Consequently the $f_{i,k}$'s need not be symmetric in all their arguments. Theorem II of the cited paper becomes true when the above changes are introduced.

In the cited paper we also need to make the following correction: substitute "It is sufficient that" for the wording "We may now apply Lemma 2 of I. D. I. V.; doing so we find" immediately preceding equation (12).

tional equations with partial functional derivatives. Of special interest are the linear functionals in which case necessary as well as sufficient conditions for invariance are given. We further demonstrate a theorem which shows the unique rôle played by a linear functional of $y(\tau)$ and $y'(\tau)$ that admits a given arbitrary *Volterra* group of transformations

$$\delta y(x) = \left[\int_0^x H(x,s)y(s)ds\right]\delta a \tag{3}$$

with

$$H(s,s) \neq 0.$$

2. *A Sufficient Condition for Invariance.* We proceed to prove the following theorem.

THEOREM I. *A sufficient condition that a functional* $f[y(\tau_0'), y'(\tau_0'), \cdots, y^{(p)}(\tau_0')]$ *with a Volterra variation and continuous functional derivatives admit a given arbitrary Fredholm group of transformations* (1) *with a kernel* $H(x,s)$ *of type* (2) ($\eta(x)$ *being assumed* to be such that all the* $\eta^{(i)}(x)/\eta(x)$*'s are continuous in the interval* 0,1) *is that it satisfy the completely integrable functional equation with partial functional derivatives*

$$\dot{f}_y(t) = -\frac{\eta'(t)}{\eta(t)}f_{y'}(t) - \frac{\eta''(t)}{\eta(t)}f_{y''}(t) - \cdots - \frac{\eta^{(p)}(t)}{\eta(t)}f_{y^{(p)}}(t), \tag{4}$$

where $f_{y^{(i)}}(t)$ *represents the partial functional derivative of* $f[y, y', y'', \cdots, y^{(p)}]$ *with respect to* $y^{(i)}(\tau)$ *taken at the point* t, *and where*

$$\eta^{(i)}(t) \equiv \frac{d^i\eta(t)}{dt^i}.$$

* To shorten the statements of the theorems that follow in this paper we shall always assume this restriction on the $\eta(x)$ found in the Fredholm kernel (2). Moreover it is scarcely necessary to state here that we assume that the derivatives of $\eta(x)$ up to and including the pth derivative exist or else our functional equations will have no meaning.

PROOF. A necessary and sufficient condition that $f[y, y', \cdots, y^{(p)}]$ admit a given arbitrary Fredholm group (1) with a kernel $\eta(x)\psi(s)$ is that

$$\delta f[y, y', \cdots, y^{(p)}] \equiv \delta a \int_0^1 \Big[f_y(t)\eta(t)\int_0^1 \psi(s)y(s)ds + f_{y'}(t)\eta'(t)\int_0^1 \psi(s)y(s)ds + \cdots + f_{y^{(p)}}(t)\eta^{(p)}(t)\int_0^1 \psi(s)y(s)ds \Big] dt \equiv 0.$$

Hence it is sufficient that

$$\eta(t)f_y(t) + \eta'(t)f_{y'}(t) + \eta''(t)f_{y''}(t) + \cdots + \eta^{(p)}(t)f_{y^{(p)}}(t) = 0,$$

which can be written in the form (4).

Paul Lévy* has treated extensively the integrability conditions and the Cauchy problem for functional equations with partial functional derivatives in the case of functionals of two independent functions. Cauchy problems and integrability conditions for equations of type (4) involving functionals of p functions can be considered by an extension of Lévy's work.

We now consider the functional equation

$$(5) \quad f_{y_p}(t) = g_1(t)f_{y_1}(t) + g_2(t)f_{y_2}(t) + \cdots + g_{p-1}(t)f_{y_{p-1}}(t).$$

Assume that each of the p functional arguments $y_1(\tau), y_2(\tau), \cdots, y_p(\tau)$ is expressed in terms of two parameters λ and μ. Let δ_λ be the variation operator when λ changes and let δ_μ be the variation operator when μ changes. By an obvious extension of Lévy's work, it can be shown that a necessary and sufficient condition that the functional equation (5) be integrable† is that

* Cf. his book *Leçons d'Analyse Fonctionnelle*, Part II.

† Cf. Lévy's book, § 78, Part I; note at the end of Chapter I, Part II; §§ 45, Part II. Lévy assumes the existence of a solution in getting his integrability conditions. In our case, however, it is possible to get explicit solutions by a direct computation. See § 4 of this paper.

$$\delta_\mu \delta_\lambda f[y_1, y_2, \cdots, y_p] \equiv \delta_\lambda \delta_\mu f[y_1, y_2, \cdots, y_p]. \tag{6}$$

A necessary and sufficient condition* that the integrability condition (6) hold is that E_{qk} be adjoint† to $E_{kq}(k, q = 1, 2, \cdots, p)$, where E_{qk} stands for the linear functional of δy_k found in the expression for $\delta f_{y_q}(t)$.

If for $y_p(\tau) \equiv y_{0p}(\tau), f[y_1, y_2, \cdots, y_p]$ is a given arbitrary functional of $y_1, y_2, \cdots, y_{p-1}$, then the $f_{y_i}(t)$'s $(i = 1, 2, \cdots, p-1)$ are known for $y_p(\tau) \equiv y_{0p}(\tau)$ and hence the linear functionals $E_{kq}(k, q = 1, 2, \cdots, p-1)$ are known and are furthermore respectively adjoint to the E_{qk}'s. To compute the E_{pi}'s and the E_{ip}'s $(i = 1, 2, \cdots, p)$, we take the variation of both sides of (5). Doing so, we get

$$\delta f_{y_p}(t) = g_1(t)\delta f_{y_1}(t) + g_2(t)\delta f_{y_2}(t) + \cdots + g_{p-1}(t)\delta f_{y_{p-1}}(t), \tag{7}$$

an identity in $\delta y_1, \delta y_2, \cdots, \delta y_p$.

Consequently on equating terms in δy_i, we get

$$\begin{aligned} E_{pi}[\delta y_i(\tau)/t] = {} & g_1(t) E_{1i}[\delta y_i(\tau)/t] + g_2(t) E_{2i}[\delta y_i(\tau)/t] + \\ & \cdots + g_{p-1}(t) E_{p-1,i}[\delta y_i(\tau)/t], \qquad (i = 1, 2, \cdots, p). \end{aligned} \tag{8}$$

From (8) we see that the E_{pi}'s $(i = 1, 2, \cdots, p-1)$ are given in terms of known linear functionals. Hence, imposing the condition that E_{pi} be adjoint to $E_{ip}(i = 1, 2, \cdots, p-1)$, we get from (8)

$$\begin{aligned} E_{ip}[\delta y_i(\tau)/t] = {} & E_{i1}[g_1 \delta y_i/t] + E_{i2}[g_2 \delta y_i/t] + \\ & \cdots + E_{i,p-1}[g_{p-1}\delta y_i/t], \quad (i = 1, 2, \cdots, p-1). \end{aligned} \tag{9}$$

Finally in order that the integrability conditions (6) be surely satisfied we must have $E_{pp}[\delta y_p(\tau)/t]$ self-adjoint. On substituting the expression for E_{ip}, as given in (9), in the expression for E_{pp} in (8), we can see with a little computation that E_{pp} is automatically self-adjoint. Hence the functional equation (5) is completely integrable. Since p can be any positive integer, it follows that (4) is also a completely integrable equation. This completes the proof of our theorem.

* This condition can readily be found by computation.

† For a definition of adjointness see Lévy's book, § 73, Part I.

The following corollaries are immediate.

COROLLARY 1. *There is one and only one solution* $f[y, y', y'', \cdots, y^{(p)}]$ *of* (4), *regarded momentarily as a functional of* $p+1$ *independent arguments, such that for a given initial functional value of the first argument*

$$y(x) \equiv y_0(x),$$

$f[y, y', y'', \cdots, y^{(p)}]$ *has arbitrary values as a functional of its last* p *arguments* $y'(x), y''(x), \cdots, y^{(p)}(x)$.

COROLLARY 2. *There always exist functionals* $f[y(x_0'), y'(x_0'), \cdots, y^{(p)}(x_0')]$, *with a Volterra variation, admitting a given arbitrary group of Fredholm transformations* (1) *with a kernel of type* (2).

3. *Linear Functionals Continuous of Order* p; *Necessary and Sufficient Conditions.** Of special interest are the linear functionals with a Volterra variation; i. e., functionals of the form

$$f_0 + \int_0^1 f_1(t)\,y(t)\,dt + \int_0^1 f_2(t)\,y'(t)\,dt + \cdots + \int_0^1 f_{p+1}(t)\,y^{(p)}(t)\,dt, \tag{10}$$

where p denotes any positive integer or zero; and all the $f_i(t)$'s are assumed continuous functions of t in the interval $(0, 1)$. We shall demonstrate the following theorem.

THEOREM II. *A necessary and sufficient condition that a linear functional* (10) *of continuity order* p *admit a given arbitrary group of Fredholm transformations* (1) *with a kernel of type* (2)† *is that it satisfy the completely integrable functional equation*

$$f_y(t) = -\frac{\eta'(t)}{\eta(t)} f_{y'}(t) - \frac{\eta''(t)}{\eta(t)} f_{y''}(t) - \cdots - \frac{\eta^{(p)}(t)}{\eta(t)} f_{y^{(p)}}(t) + \frac{\omega(t)}{\eta(t)}, \tag{11}$$

* Cf. G. C. Evans, *The Cambridge Colloquium*, 1916, Lecture III, § 3.

† We further restrict ourselves in this theorem to kernels of type (2) with $\eta(x) \neq 0$ in the interval $(0, 1)$.

where $\omega(t)$ is a given arbitrary continuous function satisfying the equation

$$\int_0^1 \omega(t)\,dt = 0. \tag{12}$$

PROOF. Evidently a necessary and sufficient condition for invariance is that

$$\left[\int_0^1 y(s)\,\psi(s)\,ds\right]\left[\int_0^1 \{f_y(t)\eta(t)+f_{y'}(t)\eta'(t)+\cdots+f_{y^{(p)}}(t)\eta^{(p)}(t)\}\,dt\right] \equiv 0$$

in y. Hence it follows, since $\psi(s) \not\equiv 0$* and since all the functional derivatives of $f[y, y', y'', \cdots, y^{(p)}]$ are point functions, that

$$\int_0^1 \{f_y(t)\eta(t)+f_{y'}(t)\eta'(t)+\cdots+f_{y^{(p)}}(t)\eta^{(p)}(t)\}\,dt = 0,$$

a condition which evidently can be expressed in the form of the functional equation (11)†.

The following important theorem is immediate.

THEOREM III. *Given any continuous function $\omega(t)$ such that* (12) *holds, and given the initial conditions in the Cauchy problem for equation* (11), *then there is one and only one linear functional* (10) *of continuity order p admitting a given arbitrary Fredholm group of transformations* (1) *with a kernel of type* (2) *and taking on the given initial conditions.*

We remark here that since $\psi(s)$ does not enter into the final conditions for invariance for the linear as well as for the non-linear functional, there is an infinitude of Fredholm groups (1) with a kernel of type (2) leaving one functional, of the types considered, invariant.

* For if $\psi(s) \equiv 0$, then our transformation degenerates into the identical one and hence there is no problem.

† Equation (11) can readily be shown to be completely integrable by a slight modification of the reasoning involved in treating equation (4).

4. *Explicit Expressions for the Functional Invariants.* To get explicit expressions for the functional invariants, we can assume $f[y, y', y'', \cdots, y^{(p)}]$ to be expansible in a series of functionals. For instance we can assume $f[y, y', y'', \cdots, y^{(p)}]$ to be an analytic functional of its $p+1$ functional arguments $y(x), y'(x), y''(x), \cdots, y^{(p)}(x)$ and then by a reasoning similar to that found in one* of the writer's papers, recurrence formulas can be computed yielding the desired functional invariants in terms of the given initial conditions. For the sake of brevity, we here give only a simple illustration.

Consider the problem of finding the functional $f[y(x_0'), y'(x_0'), y''(x_0')]$ of the form

$$
\begin{aligned}
&\int_0^1 f_{100}(t)y(t)\,dt + \int_0^1 f_{010}(t)y'(t)\,dt + \int_0^1 f_{001}(t)y''(t)\,dt \\
&\quad + \frac{1}{2!}\Big[\int_0^1\int_0^1 f_{200}(t_1,t_2)y(t_1)y(t_2)\,dt_1\,dt_2 \\
&\quad + \int_0^1\int_0^1 f_{020}(t_1,t_2)y'(t_1)y'(t_2)\,dt_1\,dt_2 \\
&\quad + \int_0^1\int_0^1 f_{002}(t_1,t_2)y''(t_1)y''(t_2)\,dt_1\,dt_2 \\
&\quad + 2\int_0^1\int_0^1 f_{110}(t_1,t_2)y(t_1)y'(t_2)\,dt_1\,dt_2 \\
&\quad + 2\int_0^1\int_0^1 f_{101}(t_1,t_2)y(t_1)y''(t_2)\,dt_1\,dt_2 \\
&\quad + 2\int_0^1\int_0^1 f_{011}(t_1,t_2)y'(t_1)y''(t_2)\,dt_1\,dt_2\Big]
\end{aligned}
\tag{13}
$$

admitting a given arbitrary Fredholm group of transformations (1) with a kernel of type (2).

We can, without any loss of generality, assume $f_{200}(t_1,t_2)$, $f_{020}(t_1,t_2)$ and $f_{002}(t_1,t_2)$ to be symmetric in t_1 and t_2. We further assume that all the f_{ikl}'s are continuous functions of all their arguments.

* *Integro-differential expressions invariant under Volterra's group of transformations*, ANNALS OF MATHEMATICS, vol. 26, p. 181 (March, 1925).

From our previous theory we see that for invariance it is sufficient that $f[y(\tau_0'), y'(\tau_0'), y''(\tau_0')]$ of type (13) satisfy the completely integrable functional equation

$$f_y(t) = -\frac{\eta'(t)}{\eta(t)} f_{y'}(t) - \frac{\eta''(t)}{\eta(t)} f_{y''}(t). \tag{14}$$

Assume the initial conditions for this equation to be given for

$$y(\tau) \equiv 0$$

in the form

$$\begin{aligned} f[0, y'(\tau_0'), y''(\tau_0')] \equiv & \int_0^1 f_{010}(t) y'(t)\, dt + \int_0^1 f_{001}(t) y''(t)\, dt \\ & + \frac{1}{2!}\left[\int_0^1 \int_0^1 f_{020}(t_1, t_2) y'(t_1) y'(t_2)\, dt_1\, dt_2\right. \\ & + \int_0^1 \int_0^1 f_{002}(t_1, t_2) y''(t_1) y''(t_2)\, dt_1\, dt_2 \\ & \left. + 2\int_0^1 \int_0^1 f_{011}(t_1, t_2) y'(t_1) y''(t_2)\, dt_1\, dt_2\right]. \end{aligned} \tag{15}$$

Calculating the partial functional derivatives of $f[y(\tau_0'), y'(\tau_0'), y''(\tau_0')]$ and substituting them in the functional equation (14) we get, on equating coefficients of similar terms in y, y', and y'', the unique determination of the four unknown f_{ikl}'s in terms of the five initially known f_{ikl}'s. Thus we get after computation

$$f_{100}(t) = -\frac{\eta'(t)}{\eta(t)} f_{010}(t) - \frac{\eta''(t)}{\eta(t)} f_{001}(t),$$

$$f_{110}(t_1, t_2) = -\frac{\eta'(t_1)}{\eta(t_1)} f_{020}(t_1, t_2) - \frac{\eta''(t_1)}{\eta(t_1)} f_{011}(t_2, t_1),$$

$$f_{101}(t_1, t_2) = -\frac{\eta'(t_1)}{\eta(t_1)} f_{011}(t_1, t_2) - \frac{\eta''(t_1)}{\eta(t_1)} f_{002}(t_1, t_2),$$

$$\begin{aligned} f_{200}(t_1, t_2) = & \frac{\eta'(t_1)}{\eta(t_1)} \frac{\eta'(t_2)}{\eta(t_2)} f_{020}(t_2, t_1) \\ & + \frac{\eta'(t_1)}{\eta(t_1)} \frac{\eta''(t_2)}{\eta(t_2)} f_{011}(t_1, t_2) + \frac{\eta'(t_2)}{\eta(t_2)} \frac{\eta''(t_1)}{\eta(t_1)} f_{011}(t_2, t_1) \\ & + \frac{\eta''(t_1)}{\eta(t_1)} \frac{\eta''(t_2)}{\eta(t_2)} f_{002}(t_2, t_1). \end{aligned}$$

5. *Functional Invariants of Volterra Groups.* It is of interest at this point to give certain results as regards linear functionals of continuity order p of type (10) which admit a given arbitrary one-parameter Volterra group of infinitesimal transformations.

The author has treated extensively in another paper* the problem of finding functionals $f[y(x_0'), y'(x_0')]$ which admit a given arbitrary Volterra group of transformations defined by (3).

In particular it was shown that a necessary and sufficient condition that a linear functional $f[y(x_0'), y'(x_0')]$ of the type

$$f_0 + \int_0^1 f_1(t)y(t)dt + \int_0^1 f_2(t)y'(t)dt \tag{16}$$

($f_1(t)$ and $f_2(t)$ assumed continuous in $(0,1)$) admit a given arbitrary Volterra group of transformations (3) with $H(s,s) \neq 0$ is that it satisfy the completely integrable functional equation

$$f_{y'}(s) = \int_0^1 L(s,r)f_y(r), \tag{17}$$

where

$$L(s,r) = \int_s^r k(s,r')\frac{H(r,r')}{H(r',r')}dr' - \frac{H(r,s)}{H(s,s)}, \quad r \geqq s,$$
$$= 0, \qquad r < s,$$

and where $k(s,r')$ is the reciprocal kernel of

$$-\frac{1}{H(s,s)}\frac{\partial H(r',s)}{\partial r'}.$$

It was also shown that there are no linear functionals

$$f_0 + \int_0^1 f(t)y(t)dt$$

($f(t)$ assumed continuous in $(0,1)$) other than constants, which admit a given arbitrary Volterra group of transformations (3) with $H(s,s) \neq 0$.

With these facts in mind we proceed to prove the following theorem.

* *Integro-differential expressions*, loc. cit.

THEOREM IV. *The linear functionals* (16) *are the only linear functionals* (10) *other than constants that can admit a given arbitrary Volterra group of transformations** (3) *with* $H(s,s) \neq 0$.

The proof of this theorem can be made to depend on the following lemma.

LEMMA. *Let* $F_0(x), F_1(x), \cdots, F_k(x)$ *be continuous functions in the interval* $(0,1)$. *If*

$$(18)\qquad \int_0^1 [F_0(x)y(x) + F_1(x)y'(x) + \cdots + F_k(x)y^{(k)}(x)]dx = 0$$

for all possible forms of the function $y(x)$, *having a continuous* k*th derivative* $y^{(k)}(x)$ *in* $(0,1)$, *then it is necessary*† *that* $F_k'(x)$ *exist and be continuous in* $(0,1)$ *and that* $F_k(0) = F_k(1) = 0$.

PROOF. Assume

$$(19)\qquad y(0) = y(1) = y'(0) = y'(1) = \cdots = y^{(k-1)}(0) = y^{(k-1)}(1) = 0.$$

Then making use of a well known formula‡ and by an evident use of Dirichlet's formula, we get

$$(20)\qquad \int_0^1 y^{(k)}(t)\left[F_k(t) + \int_t^1 \left\{F_{k-1}(x) + F_{k-2}(x)(x-t) + \cdots + F_1(x)\frac{(x-t)^{k-2}}{(k-2)!} + F_0(x)\frac{(x-t)^{k-1}}{(k-1)!}\right\}dx\right]dt = 0$$

for all y of our hypotheses satisfying (19). It follows§ therefore that

* We note, without any further statement, that whenever we shall need the pth extended Volterra group of transformations, we shall assume that all the derivatives of $H(x,s)$ and $H(s,s)$ that are found in this extended transformation exist and are continuous.

† The reader can, without difficulty, push the argument until necessary and sufficient conditions on all the $F_i(t)$'s are found by successive applications of our lemma and by successive valid integrations by parts; but the lemma as stated is sufficient for our purpose.

‡ Cf. Hadamard, *Calcul des Variations*, vol. I, p. 201.

§ Cf. Hadamard, loc. cit.

$$F_k(t) = -\int_t^1 \Big\{F_{k-1}(x) + F_{k-2}(x)\,(x-t) +$$

$$(21) \quad \cdots + F_1(x)\frac{(x-t)^{k-2}}{(k-2)!} + F_0(x)\frac{(x-t)^{k-1}}{(k-1)!}\Big\}dx$$

$$+ l_0 + l_1(1-t) + l_2(1-t)^2 + \cdots + l_{k-1}(1-t)^{k-1},$$

where the l_i's are arbitrary constants. The derivative of the right hand side of (21) exists and is continuous in $(0,1)$ and hence the derivative $F_k'(t)$ of $F_k(t)$ exists and is continuous in $(0,1)$. Dropping now the restrictions (19) and calculating $F_k'(t)$ from (21) we proceed as follows. We substitute the expression for $F_k(t)$ in (18) and perform the now valid integration by parts. Carrying out $k-1$ valid successive integrations by parts starting with the term involving $y^{(k-1)}(t)$ and ending with the resultant term involving $y'(t)$, we get, on putting

$$(22) \quad y(0) = y(1) = y'(0) = y'(1) = \cdots = y^{(k-2)}(0) = y^{(k-2)}(1) = 0,$$

$$(23) \quad [F_k(t)y^{(k-1)}(t)]_0^1 = 0$$

for all y's satisfying (22). In order that (23) may hold it is necessary and sufficient that $F_k(0) = F_k(1) = 0$. This completes the proof of our lemma.

A necessary and sufficient condition that a linear functional (10) admit a given arbitrary Volterra group of transformations (3) is that

$$\int_0^1 [y(t)h_1(t) + y'(t)h_2(t) +$$

$$\cdots + y^{(p-2)}(t)h_{p-1}(t) + y^{(p-1)}(t)H(t,t)f_{p+1}(t)]dt \equiv 0$$

in y, where $h_i(t)$ $(i = 1, 2, \cdots, p-1)$ are easily calculated expressions whose explicit form is not necessary to the argument. On applying the lemma just proved we conclude that it is necessary that the derivative of $H(t,t)f_{p+1}(t)$ exist and be continuous and that

$$H(1,1)f_{p+1}(1) = H(0,0)f_{p+1}(0) = 0.$$

Since by hypothesis $H(t,t) \neq 0$, we conclude that the derivative of $f_{p+1}(t)$ exists and is continuous and that $f_{p+1}(1) = f_{p+1}(0) = 0$. Hence it is necessary that the linear functional of continuity order p reduce, by an integration by parts, to a linear functional of continuity order $p-1$. The lemma applied again to this new functional reduces it to a linear functional of continuity order $p-2$. Applying the lemma successively in such a manner, we finally get, as a necessary condition for invariance, the result that the original linear functional of continuity order p has to reduce at least to a linear functional of continuity order one. This result coupled with the existence theorems cited in the beginning of § 5, establishes our theorem.

THE RICE INSTITUTE

ON THE DISTRIBUTION OF QUADRATIC AND HIGHER RESIDUES*

BY H. S. VANDIVER

1. *Introduction.* In the present paper theorems will be obtained regarding the distribution of quadratic and higher residues. Special cases of the theorems yield results concerning the class number of quadratic forms of determinant $(-d)$ where $d \equiv 3 \pmod 4$.

2. *Conjugate Sets of Residues.* In a previous article† the writer considered the notion of *conjugate set* in a *finite algebra.* Applied to the finite algebra represented by the residue classes of a rational integer as modulus, we may define a *conjugate set of residues* of the modulus m to be a set

$$a_1, a_2, \cdots, a_i \tag{1}$$

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* Presented to the Society, October 28, 1916.

† ANNALS OF MATHEMATICS, (2), vol. 18 (1917), p. 106.

which has the property that

$$na_1,\ na_2,\ \cdots,\ na_i$$

may be expressed, modulo m, as a permutation of the original set, n being some rational integer $\not\equiv 1 \pmod{m}$. In this case n is called a *multiplier* of the set. Let r, s, and $(r+s)$ all be multipliers of the set (1), modulo $m=p$, a prime, and $0<a_c<p$, $(c=1, 2, \cdots, i)$. Consider the expression

$$\sum_h \frac{|rh|+|sh|-|(r+s)h|}{p} h^{kp},$$

where h ranges over the elements of (1), and $|x|$ is the least positive residue of x, modulo p. Then

$$rh = |rh|+pt,$$

where t is an integer, and

$$(2)\qquad \begin{cases} (rh)^{pk} \equiv |rh|^{pk} & \pmod{p^2}, \\ h^{pk} \equiv r^{p(p-1)-pk}|rh|^{pk} & \pmod{p^2}. \end{cases}$$

Similarly

$$(3)\qquad h^{pk} \equiv s^{p(p-1)-pk}|sh|^{pk} \pmod{p^2},$$

$$(4)\qquad h^{pk} = (r+s)^{p(p-1)-pk}|(r+s)h|^{pk} \pmod{p^2}.$$

Since r is a multiplier of (1), we have $\sum_h |rh| = \sum_{c=1}^{i} a_c$; and similarly, for $\sum_h |sh|$ and $\sum_h |(r+s)h|$. Hence

$$\sum_h (|rh|h^{pk}+|sh|h^{pk}-|(r+s)h|^{pk}) \equiv (r^e+s^e-(r+s)^e)\sum_{c=1}^{i} a_c^{pk+1} \pmod{p^2},$$

or

$$\sum_h \frac{(|rh|+|sh|-|(r+s)h|)h^{pk}}{p} \equiv \frac{(r^e+s^e-(r+s)^e)\sum_{c=1}^{i} a_c^{pk+1}}{p} \pmod{p},$$

where $e=p(p-1)-pk$. The coefficient of h^{pk} on the left is 0 or 1 according as $|rh|+|sh|$ is less than or

greater than p. Hence we have proved the following theorem.

THEOREM I. *If $a_1, a_2, \cdots, a_i$ is a conjugate set modulo p, and $|x|$ is the least positive residue of x modulo p, then*

$$\sum_{h_1} h_1^k \equiv \frac{(r^e+s^e-(r+s)^e)\sum_{c=1}^{i} a_c^{pk+1}}{p} \pmod{p}, \tag{5}$$

where h_1 ranges over the a's which satisfy $|rh_1|+|sh_1|>p$ and $e=p(p-1)-pk$.

3. *The Number of j-ic Residues.* In Theorem I, let $a_1, a_2, \cdots, a_i$ be the i roots of $x^i \equiv 1 \pmod{p}$, $p=ij+1$; and suppose that r, s, and $(r+s)$ are each congruent, mod p, to an integer included among the a's. Set $k=i$. Then (2) gives

$$H \equiv \frac{\sum_{c=1}^{i} a^c}{p} \pmod{p},$$

where H is the number of j-ic residues included in the integers $h<p$ such that $|rh|+|sh|>p$. Each side of this congruence is a positive integer less than p; consequently

$$H = \frac{\sum_{c=1}^{i} a_c}{p};$$

hence we have the following theorem.

THEOREM II. *If $p=ij+1$ and $a_1, a_2, \cdots, a_i$, $(0<a_c<p$, $c=1, 2, \cdots, i)$, are the incongruent roots of $x^i \equiv 1 \pmod{p}$, r, s, and $(r+s)$ being j-ic residues of p, then*

$$H = \frac{\sum_{c=1}^{i} a_c}{p},$$

where H is the number of j-ic residues included in the integers $h<p$ whi h satisfy $|rh|+|sh|>p$.

This result was previously known for the case $r=1$, $s=1$, $j=2$.

4. *Generalization for Quadratic Residues.* For quadratic residues we shall obtain a more general theorem. Let the set (1) range over the distinct positive quadratic residues of p which are less than p, and suppose now that r and s are not necessarily quadratic residues of p. Then if $(r/p)=1$,

$$\sum_h |rh| = \sum a.$$

If however $(r/p) = -1$, then

$$\sum_h |rh| = \sum b = p(p-1)/2 - \sum a,$$

where b ranges over the distinct quadratic non-residues $0<b<p$. Each of these relations is included in the statement

$$\sum_h |rh| = \left(1-\left(\frac{r}{p}\right)\right)p\mu + \left(\frac{r}{p}\right)\sum a,$$

where $\mu = (p-1)/4$, (not necessarily an integer), and r is any integer prime to p. If we take s and $(s+r)$ in lieu of this relation we get two other equations, which, with the original, give

$$\sum_h \frac{|rh|+|sh|-|(r+s)h|}{p} = (1-R)u + \frac{R\sum a}{p},$$

$$R = \left(\frac{r}{p}\right)+\left(\frac{s}{p}\right)-\left(\frac{r+s}{p}\right).$$

A term in the expression on the left of (6) is one or zero according as $|rh|+|sh|$ greater than or less than zero. We may then state the following theorem.

THEOREM III. *If p is an odd prime and K is the number of quadratic residues h included in the set $1, 2, \cdots, p-1$, and satisfying $|rh|+|sh| > p$, then*

$$K = (1-R)\mu + R\frac{\sum a}{p}$$

where $\mu = (p-1)/4$, and

$$R = \left(\frac{r}{p}\right)+\left(\frac{s}{p}\right)-\left(\frac{r+s}{p}\right),$$

r, s, and $(r+s)$ being prime to p, with the a's ranging over the positive quadratic residues of p which are less than p.

For the case $s=1$, this theorem was previously stated in another form and without proof by the writer*. For $s=r=1$, the relation reduces to an equality which is well known.

We shall now show that the number of distinct integers h in the set $1, 2, \cdots, p-1$, which satisfy $|rh|+|sh|>p$ are $(p-1)/2$ in number. If $|rh|+|sh|<p$ then

$$rh = i_1 p+\varepsilon_1,\ |sh| = i_2 p+\varepsilon_2,$$

ε_1 and ε_2 being each less than p. Also

$$(p-h)r = pr-rh = p(r-i_1)-\varepsilon_1,$$

and

$$|(p-h)r| = p-\varepsilon_1.$$

Similarly,

$$|(p-h)s| = p-\varepsilon_2;$$

hence

$$|(p-h)r|+|(p-h)s| = 2p-\varepsilon_1-\varepsilon_2.$$

Also, by definition, $|rh| = \varepsilon_1$, $|sh| = \varepsilon_2$ and $\varepsilon_1+\varepsilon_2<p$, therefore $2p-\varepsilon_1-\varepsilon_2>p$. Hence if h satisfies $|rh|+|sh|>p$, then $p-h$ will not. This shows that there are $(p-1)/2$ distinct values h.

5. *Conclusion.* From Theorem III we see that K is immediately determined if $\sum a/p$ is known, since, for $p=4n+3$,

$$2\left(1-\left(\frac{2}{p}\right)\frac{1}{2}\right)\frac{\sum b-\sum a}{p} = h(-p)$$

is the expression for the number of classes of properly primitive quadratic forms of determinant $(-p)$. Since also $\sum b+\sum a = (p^2-p)/2$, it is possible to give a relation between h and K. In particular we obtain a variety of expressions for h.

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* This BULLETIN, vol. 23 (1916), p. 113.

COOLIDGE ON COMPLEX GEOMETRY

The Geometry of the Complex Domain. By Julian Lowell Coolidge. New York, Oxford University Press, American Branch, 1924. 242 pp.

The various geometries of a complex space are, with certain definite exceptions, essentially the same as the corresponding geometries of a real space, provided that one agrees, tacitly or wittingly, to admit to consideration only those configurations in complex space which are the direct generalizations of configurations in real space in that they arise from these by the replacement of real by complex numbers. This agreement restricts, for example, the discussion of manifolds of complex points to those manifolds depending on a fixed number of complex parameters. But a system of points depending on n complex parameters is but a very special case of a system depending on the equivalent number, $2n$, of real parameters, and its properties are in no way indicative of those of the general $2n$-parameter system. In other words, when the scope of complex geometry is widened to cover all possibilities, it takes on an entirely different aspect from that of real geometry.

Though there are recent books on complex geometry which pursue the subject along the narrow path outlined by the geometry of reals, the one now under review is the first to follow consistently the wider and, for the progress of mathematics, the more important, point of view. It aims to bring the reader abreast of the times in the advances, from this point of view, of the last half-century. The pioneers in these researches have been Segre and Study, and to them the book is informally dedicated. Among their disciples are to be mentioned Autonne, Benedetti, Coolidge, Fubini, Loewy, Sforza, and J. W. Young.

Closely connected with, if not strictly speaking a part of, complex geometry is the classical problem of representing complex points by real elements. An historical and critical account of the many attempts, ancient and modern, successful and unsuccessful, to solve this problem forms the second integral part of the book. The writers here are legion; we content ourselves with the names of the more important: Wallis, Wessel, Argand, Gauss, Poncelet, von Staudt, Laguerre, Marie, Klein, Segre, and Study.

The material has been well digested and excellently ordered. The complex line (Chapters I, II), the complex plane (Chapters III–VI), and complex three-dimensional space (Chapter VII) are taken up in turn. In each case, the real representations of complex points are first considered and then later put to work in throwing light on the complex geometry itself. The author writes in his characteristic, red-blooded style; his historical accounts and criticisms are, in particular, enter-

taining and stimulating. He has at various stages, notably in connection with the differential geometry of manifolds of points, contributed original results.

The developments of the wider aspects of complex geometry pertain largely to manifolds of points depending on a fixed number of real parameters. The study of algebraic manifolds and their projective properties has received the lion's share of attention and forms a dominant feature of the present book. The general analytic manifolds and their differential properties have been discussed only recently, by the author himself.

If $x_1, \cdots, x_n, x_{n+1}$ are complex variables and $\bar{x}_1, \cdots, \bar{x}_n, \bar{x}_{n+1}$ are their conjugates, a polynomial which is homogeneous and of degree m in each set of variables and has the property that it is a constant multiple of the conjugate-complex polynomial, is called a hyperalgebraic form. Using symbolic notation, we can write: $(ax)^m(\bar{a}\bar{x})^m$, where $(ax) = a_1x_1 + \cdots + a_nx_n + a_{n+1}x_{n+1}$. When the order, $2m$, of the form is equal to two, it reduces to the better known Hermitian form, $(ax)(\bar{a}\bar{x})$.

If we think of $x_1, \cdots, x_n, x_{n+1}$ as homogeneous coordinates of a point in a complex space of n dimensions, and set our hyperalgebraic form equal to zero, the locus of the equation obtained, if it has any locus other than singular points, is an algebraic manifold of the type just mentioned. The number of real parameters on which it depends is $2n-1$, one less than the total number of real degrees of freedom of a point in the space in question. Manifolds coming under this head are the predominant algebraic manifolds in their respective spaces; among them, the simplest and most interesting are those represented by Hermitian forms.

With this orientation concerning the general scope of the book, we can proceed with greater appreciation to a detailed discussion of its contents.

The first chapter deals with the attempts to represent by real elements the complex points of a line. Outstanding is, of course, the classical representation of analysis, first discovered by Wessel and later rediscovered by Gauss and Argand. It is interesting to note that it was Gauss who first introduced the representation by means of points; Wessel and Argand contented themselves with vectors.

The projective geometry of the complex line is the image of the real geometry of inversion in the Gauss plane, and we can best outline the content of the second chapter from this point of view. To a circle or a straight line in the Gauss plane corresponds on the complex line a system of points depending on one real parameter, known as a chain. A chain is characterized, analytically, by the fact that it can be represented by a Hermitian form, and, geometrically, by the

properties that the cross ratios of any four of its points are real and that it contains the point making with any three of its points any given real cross ratio. The direct circular transformations of the Gauss plane correspond to the collineations of the line, whereas the indirect circular transformations, when interpreted on the line, are called anti-collineations. Collineations preserve a given cross ratio, anti-collineations change it to the conjugate-complex value; both carry a chain into a chain, and, taken together, they are the only continuous one-to-one transformations of the complex line which preserve harmonic separation.

Among the many further developments of this chapter we mention only the study of algebraic threads. A thread is a system of points depending on one real parameter. An algebraic thread can be represented by a hyperalgebraic form. It corresponds in the Gauss plane to an algebraic curve, of order equal to that of the form. The same order is given to the thread and a geometrical significance is found for it. Class is also defined for the thread, by the application of a transformation on the line corresponding to the polar reciprocation in the circle $x^2+y^2+1=0$ in the plane.

In the projective geometry of the complex plane (Chapter V), we have, besides the chain of points, its dual, the chain of lines; the two are projectively equivalent, respectively, to the range of real points and the pencil of real lines. The projective equivalents of the real points (lines) of the plane play also fundamental roles; these are manifolds of points (lines) depending on two real parameters, known as chain congruences of points (lines). As in the previous case, we have here both collineations and anti-collineations.

Just as ordinary polar systems lead to conics, the anti-polarities yield the so-called hyperconics. These are represented by Hermitian forms. They consist, then, not of ∞^2 but of ∞^3 points. On the other hand, there are no curves which are entirely contained in one of them. A straight line meets one either in a chain of points, a single point, or nowhere; the case of intersection in a single point occurs only when the line contains its pole in the given anti-polarity, and so such a line is called a tangent. Discussion of further developments, of the theory of linear systems of hyperconics, and of the work on hypercurves, three-parameter systems of points represented by hyperalgebraic forms, we must forego.

The geometry of the subgroup of the group of collineations which possesses the Hermitian form $(x\bar{x})$ as a relative invariant is known as a Hermitian metric of elliptic type. This metric and the parabolic metric resulting from it in the usual way are developed in detail. In the elliptic case, to which we confine ourselves, the governing group depends on three complex and three real parameters. Distance and

angle are always real. A geodesic thread turns out to be a particular type of chain, characterized by the fact that its points are, in pairs, conjugate with respect to the fundamental form. The Hermitian trigonometry of a triangle is found to be identical with that of ordinary elliptic trigonometry only in case the altitudes are concurrent. In the discussion of the metric properties of hyperconics, it is discovered that the general hyperconic has a whole chain of foci and a chain of corresponding directrices, and that the ratio of the sines of the distances of a point on the hyperconic to a focus and to the corresponding directrix is not merely constant, but the same for all foci!

A representation of the complex points of the plane which is peculiarly applicable to the projective and metric theories just reviewed, is that of Segre. Thereby, each point is represented by a real point of a four-dimensional variety of the sixth order in a space of eight dimensions. Through each point of this S_4^6 pass two conjugate imaginary planes of the S_4^6. The collineations and anti-collineations of the plane are represented in S_8 by real collineations, which, in the one case, permute the planes of each of these systems and, in the other, interchange the two systems. A straight line is represented by a quadric surface, a chain by a conic, a chain congruence by a surface of Veronese (Cayley), and a hyperconic by the intersection of the S_4^6 by a hyperplane. The Hermitian metrics appear on S_4^6 as classical non-euclidean metrics!

Chapters III and IV are devoted to a review of the various methods of representing the complex points on a curve and in a plane, respectively. Aside from the representation of Segre, just described, and that of Klein and Study, to which, for lack of space, we can give but passing mention, the most important of these methods are those of von Staudt, Marie, and Laguerre. According to von Staudt, an imaginary point is represented by the elliptic involution on the real line through the imaginary point which has as its double points the imaginary point and its conjugate, or, more accurately, by this involution provided with a sense. The Laguerre representation consists of the two real points, taken in a specified order, in which the isotropic lines through the given imaginary point intersect those through the conjugate-imaginary point. The Marie representation is also an ordered pair of points, lying on the real line through the given imaginary point. These three representations of an imaginary point, though apparently essentially different, are closely related. The points of the Laguerre representation, when rotated through a right angle about the mid-point of their segment, become those of the Marie representation; these are, in turn, a pair of points in the von Staudt involution, in fact, that pair whose distance apart is least.

The Laguerre and Marie representations of a curve have been exhaustively treated by Study. They both consist, in general, of real

transformations of the plane, inversely conformal transformations in the one case, and special directly equi-areal transformations in the other. These results are developed in Chapter VI, on the differential geometry of the plane. The remainder of the chapter consists of the author's recently published work on the differential geometry of two- and three-parameter manifolds of points.

The geometry of three-dimensional complex space is considered in Chapter VII. The projective theory follows, generally speaking, the lines of that in the plane. The representation of Marie still consists of an ordered pair of points; when applied to a curve, it yields two (translation) surfaces corresponding by an elliptic parallel map which is equi-areal and of invariant -1.* The Laguerre representation is not so simple, consisting now of an oriented circle; the results thus far obtained concerning it are interesting and there is still room for further labors.

An interesting feature of this chapter is the geometry of the minimal plane (Beck, C. L. E. Moore). The chapter closes with some previously unpublished work of the author dealing with analytic manifolds depending on n real parameters, $n = 2, 3, 4, 5$, and establishing in particular the conditions that such a manifold contain one or more curves or surfaces.

A book on complex geometry can hardly be considered complete unless it establishes a logical foundation for the subject. The simplest foundation is an analytical foundation, based on the complex number system, and such a foundation is inherent in the opening chapters of the book. In the last chapter, the author carries through a more adventurous method, by means of postulates. A system of postulates for the real projective domain, that of his book on Non-Euclidean Geometry, is first developed and projective coordinates for real elements introduced. Imaginary elements are then defined by means of the elliptic involutions of von Staudt and finally acquire coordinates through the medium of von Staudt throws.

The reviewer has but one serious quarrel with the author. To be sure, he would have been happier if the author had followed Study in choosing the word membrane instead of the over-worked word congruence to designate a system of points depending on two real parameters. But this is a matter of minor importance. What caused much greater mental anguish was the author's indiscriminate use of the words complex and imaginary. He seems to have made no conscious

* Cf. Graustein, *Real representations of complex curves*, Annals of Mathematics, (2), vol. 26 (1924), p. 143. The previous description by the reviewer, reproduced by Coolidge, has the disadvantage that it employs imaginary curves (the generators) on the representing surfaces.

attempt to distinguish between the two and repeatedly uses complex as the opposite of real. In a book on complex geometry this is especially to be regretted.

The book is marred by many, very many, typographical errors. Those in the text itself are not seriously disturbing; in fact, some of them, as, for example, "any value linearly descendent on two given values" on p. 108, and "two intersectional minimal lines" on p. 190, are decidedly the opposite. But those in the formulas, though they may be only irritating to the mature reader, are likely to prove a stumbling block for the budding mathematician.

Aside from these matters, the book is full of solid, stimulating mathematics. Moreover, it is particularly welcome in that it brings up to date a field of geometry which is comparatively new and by no means exhausted. May it prove the inspiration and basis of departure for fresh endeavors.

W. C. GRAUSTEIN

GAUSS AND NON-EUCLIDEAN GEOMETRY

A question of historical accuracy is raised by Professor Emch in his review of my *Projective Geometry** where he says:

"It is proper to point out an error which is common in histories of mathematics and which is contained in the following statement on page 420: 'Little progress was made until about a hundred years later when Gauss (1777–1855), his friends and pupils became deeply interested in the subject.' (1) Now the fact is that Gauss's deeper interest in the subject was subsequently aroused by the brilliant discoveries of Lobatchevsky and Bolyai. (2) As a matter of fact, Gauss, in the beginning, hoped to be able to prove what is known as the euclidean parallel axiom and (3) assumed a rather skeptical attitude towards the new discoveries. (4) Subsequent deeper meditations, however, led Gauss to his own establishment or verification and acceptance of the new theory."†

An examination of the sources on which my statement is based will, I believe, substantiate the sentence quoted as well as the context from which it is taken. Among these sources are the letters of the

* This BULLETIN, vol. 30 (1924), p. 81.

† The numbering is mine.

two Bolyais and Gauss himself.* Of the four points in Professor Emch's own statement, (2) is correct but irrelevant. For it applies equally to all the pioneers in the subject, including Lobachevsky and young Bolyai too † who may be said to have inherited the problem of parallels from his father, Wolfgang Bolyai.

In Gauss's works (loc. cit.) are found some twenty letters dealing with the foundations of geometry and the theory of parallels, the first written in 1799. From these we learn that he had attacked the problem as early as 1792 (letter to Schumacher, 1846) while he first learned of the work of John Bolyai when he received from the father a copy of the famous *Appendix* in 1832 (letter to Gerling).‡ Prior to this time he expresses his doubts concerning the validity of Euclid (letter to W. Bolyai, 1799) and states his conviction that "the necessity of our geometry cannot be proved" and hence "we must place geometry somewhat in the category with mechanics" (letter to Olbers, 1817). He obtained many of the fundamental properties of the hyperbolic geometry, e. g. the impossibility of similar figures, the existence of an absolute *a priori* measure of length (letter to Gerling, 1816), the connection between the defect of the angle sum and the area of a triangle, the formula for the area of the maximum triangle (letter to Gerling, 1819) and the formula for the length of a circle (letter to Schumacher, 1831). He even originated the name *non-euclidean geometry*, declared it to be self-consistent throughout and that it reduced to the euclidean when the space constant is infinite (letter to Taurinus, 1824). Various reasons are assigned for not publishing his results, such as the lack of time for refining them, the indifference and incompetence of the mathematical public and the anticipated "clamor of the Boeotians".

This seems to me ample justification of my statement that Gauss was deeply interested in the problem of parallels and had made sub-

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* Stäckel and Engel, *Gauss, die beiden Bolyai und die nichteuklidische Geometrie*, MATHEMATISCHE ANNALEN, vol 49 (1897), p. 149; Gauss, WERKE, vol. 8, pp. 159–239. Many of these are available in English, e. g. in Carslaw, *Non-Euclidean Geometry*, pp. 19–37.

† Three proofs of the parallel postulate by Lobachevsky are contained in notes of his lectures (1815–1817) and another apparently in a paper, now lost, but read at Kasan in 1826, entitled *Exposition succincte des principes de la géometrie, avec une démonstration rigoreuse du théorème des parallèles.*—Carslaw, loc. cit. John Bolyai was seeking a proof of the postulate at about 1820 according to a statement in his autobiography, quoted by Stäckel and Engel, loc. cit., p. 155.

‡ He appears not to have heard of the work of Lobachevsky until some years later (letter to Encke, 1841) and notes by Stäckel.

stantial progress toward its solution prior to his knowledge of the work of J. Bolyai and Lobachevsky. It would be interesting, if space permitted, to listen at length to Gauss's own testimony but a few extracts from his letters will have to suffice:

1. To Schumacher, May 17, 1831. "Some of my own meditations which in part are now about 40 years old—whereof however I have never recorded anything and therefore have been obliged to think out many of them three or four times anew—I have begun in the past few weeks to write down. For I did not wish it to perish with me."

2. To Gerling, February 14, 1832. ... "I have this day received from Hungary a little treatise on the Non-Euclidean geometry wherein I find again all my own ideas and results, developed with great elegance, howbeit for one to whom the subject is strange, in a form somewhat difficult to follow because of the concentration. ... I regard this young geometer von Bolyai as a genius of the first order."

3. To Wolfgang Bolyai, March 6, 1832 (in response to a letter accompanying the *Appendix* by John Bolyai). ... "If I should begin by saying that I am unable to praise it, you would doubtless be astonished for a moment. But I cannot do otherwise—to praise it would be to praise myself. For the whole content of the work, the method which your son has employed and the results to which he is led, agree almost completely with my own meditations pursued in part now for 30–35 years." ...*

4. To Schumacher, November 28, 1846. ... "I have recently had occasion to examine again the little work by Lobatschewsky (*Geometric Researches on the Theory of Parallels*, Berlin, 1840). It contains the elements of that geometry which must exist and with strict consistency may exist if the euclidean geometry is not the true one. A certain Schweikart called such a geometry *astral geometry*, Lobatschewsky *imaginary geometry*. You know that for 54 years (since 1792) I have held the same conviction (with a certain later extension which I shall not mention here). I have not found therefore in Lobatchevsky's work material new to me but the development is made on other lines than I have followed and indeed by Lobatchevsky in amasterly manner with true geometric genius." ...

R. M. WINGER

* John Bolyai was only 29 years old when this was written.

JULIA ON ESSENTIAL SINGULARITIES

Leçons sur les Fonctions Uniformes à Point Singulier Essentiel Isolé. By Gaston Julia. Redigées par P. Flamant. (Borel Monograph.) Paris, Gauthier-Villars, 1924. vii + 149 pp.

This monograph deals with Picard's famous theorem of 1879 to the effect that an analytic function, in the neighborhood of an isolated essential singularity, assumes every value, with two possible exceptions, an infinite number of times. It presents this theorem, and perfections of it due to Landau, Schottky, Caratheodory, Lindelöf, Iversen, and to Julia himself.

A first chapter develops as much of the theory of modular functions as is needed for the proof of Picard's theorem. The theorem that any simply connected region can be mapped conformally upon a circle is here used, and familiarity with it by the reader is assumed. Of course, an elementary proof of Picard's theorem, by the methods of Borel, Landau and Schottky, would not require as much preliminary work, but when one considers the intuitive qualities of Picard's own proof, one can understand Julia's preference. Besides, as Caratheodory has shown, (this is dealt with in the second chapter), the modular function does not play an artificial rôle, for it leads to the determination of an important least upper bound, in the expression for which the modular function actually appears.

The second chapter opens with the proof of Picard's theorem for the special case of an integral function. Then comes the following generalization, due to Landau:

There exists a function $\varphi\ (a_0, a_1) > 0$, *such that, if the series*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots, \tag{1}$$

with $a_1 \neq 0$, *converges for* $|x| < \varphi(a_0, a_1)$, *the series assumes at least one of the values* 0 *or* 1 *for* $|x| < \varphi\ (a_0, a_1)$.

Caratheodory's determination of the best $\varphi(a_0, a_1)$ is then given. As stated above, the modular function appears in this expression. The next theorem is that of Schottky:

There exists a function $M\ (a_0, \theta)$, *such that, if* (1) *converges for* $|x| \leqq R$ *and does not assume either of the values* 0 *or* 1 *for* $|x| \leqq R$, *we have, for any positive* θ *less than unity, and for* $|x| < \theta R$, *the inequality* $|f(x)| < M\ (a_0, \theta)$.

The chapter closes with the proof of Picard's theorem for any isolated essential singularity.

The third chapter, one of the most interesting in the book, gives an account of Montel's *normal families* of functions, which Julia used

in his own researches. A set of functions, meromorphic in an area, is said to be *normal* in the area if from every sequence of functions of the family a sub-sequence can be extracted which converges uniformly in every area interior to the given one. Here the notion of uniform convergence has to be loosened somewhat, because of the poles. Also, the infinite constant is an allowable limit for the sub-sequence.

Montel's chief theorem states that *a sufficient condition for a family to be normal in an area is that three distinct values, a, b and c exist, none of which is assumed in the area by any function of the family.*

Montel's work on normal families has been singularly fruitful in many connections. One might mention conformal mapping, iteration and Picard's theorem. Still it seems to the present reviewer that Montel's principal theorem is little more than a combined statement of Schottky's theorem and of well known results on equally continuous families of functions.

An interesting detail of the third chapter is the crediting of M. B. Porter with the independent discovery of the theorem on sequences of analytic functions usually known as Vitali's theorem.

Chapter IV deals with researches of the Scandinavian school of analysts on sectors with vertex at an essential singularity. We pass over these results.

We come now to Julia's own work, which occupies the last three chapters. Julia's first problem amounts to this: *If $f(z)$ is a meromorphic function, and if $|\sigma| > 1$, what can be said as to the values assumed by the sequence of functions $f(\sigma^n z)$, $n = 1, 2, \cdots$, in any given area?* Those familiar with the recent work on the iteration of rational functions will see a relation between this problem and the theory of Poincaré's functions with rational multiplication theorems, which functions were introduced into iteration theory by S. Lattés and by the present reviewer. Julia's work deals principally with the set of points at which the family $f(\sigma^n z)$ is not normal. The results are numerous and interesting. In the final chapter, a set of multipliers not of the form σ^n is used. As a capital yield of Julia's methods, we cite the following extension of Picard's theorem for integral functions: *Given an integral function $f(z)$, and any point of the plane, there exists a ray joining the point to infinity such that, in any sector containing the ray in its interior, $f(z)$ assumes every finite value, with one possible exception, an infinite number of times.*

Julia's monograph is probably one of the finest in the Borel collection. The student who reads it together with Landau's *Neuere Ergebnisse der Funktionentheorie* will acquire dominion over a region of analysis whose exploitation is not yet complete.

J. F. RITT

THE N. R. C. HANDBOOK ON STATISTICS

Handbook of Mathematical Statistics. By H. L. Rietz, editor in chief, and eight other members of the Committee on the Mathematical Analysis of Statistics of the Division of Physical Sciences of the National Research Council. Boston, Houghton Mifflin Company, 1924. viii + 221 pp.

The general purposes of this book may be inferred from statements in its preface and from its form. It deals "with the mathematical analysis of data", not with its collection or interpretation. "Special emphasis is laid on the limitations surrounding the proper applications of the various methods." It is not a treatise or a text-book. It is to be used as a reference book, probably also as a background for a course of lectures. At the same time, it provides sufficient explanation of the fundamental ideas, and especially sufficient illustration of the methods, and sufficient bibliography, to enable the reader to begin his acquaintance with statistical theory here, provided he has already a knowledge of collegiate mathematics. This last proviso is, in the mind of the reviewer, a distinct advantage that this book has over some others; it is possible to make the subject so much clearer with the use of mathematical language than without it. Apparently, also, having been written by a committee of nine authors, to each of whom a piece was allotted, the book is intended to be representative of expert knowledge in the several portions of its field.

This was an obvious and decided advantage of multiple authorship, and one cannot but recognize the appropriateness of choice of author in each case. There were also some obvious disadvantages, especially as the several authors were scattered about the country from the Atlantic seaboard to the Pacific. If the book is to serve well as a basis for lectures, it is important that there should be consistency both of notation and of language throughout. This is the more desirable because at best the notation of mathematical statistics is complicated. It was a difficult thing for a group of nine to achieve, and in fact it was not very well achieved. One is led to feel that, from a practical point of view, it would have been better to have had a smaller number, or at least a different division of labor among the nine. The chapters on multiple and simple correlation ought not to be widely different from each other in style, method of approach, notation and language. There would seem to be no good reason why the subject of frequency curves should be introduced by one writer and carried on by another, or why Bernoulli, Poisson, and Lexis distributions should be handled by a different author from the one who deals with probability. As a result

there are such inconsistenties as the following. In Chapter I, page 15, moments are defined as in mechanics, not as usually in statistics. This definition is in agreement, though the notation is slightly in disagreement, with the usage by the same author in Chapter IV, page 69. In Chapter VII, however, the statistical definition is used, but the notation of Chapter I. That is, there are two different definitions with the same notation, and two different notations with the same definition. There is also the so-called "Bernoulli Theorem" in two different notations, pages 16 and 72, so that the misprint in the first statement is not immediately obvious by comparison with the second. Further, there is a confusion of statements so that the reader cannot be sure that the expressions really include areas on both sides of the mode until he has examined the illustrative example on page 73.

Another considerable difficulty with which the committee had to deal was that of securing uniform adherence to their excellent ideal of stating and emphasizing the limitations of methods. It is more important that this be done meticulously in a handbook than either in an elementary text-book or in a treatise, for, were the proofs of the theorems given, the careful reader could discover these limitations for himself. Since the proofs are not usually given the reader has a right to demand extraordinary care in the statements. It is a joy to find the many places where special care is manifest, e. g., in the statements regarding the χ test on page 80, and that generally, throughout the book, there is the spirit of mathematical rigor. There are some slips. In presenting the formulas for probable errors, for example, no author states that the parameters appearing on the right are, strictly, the parameters of the universe from which the sample is drawn, in some cases (p. 32; p. 77, probable errors of r, η, $r\ \sigma_y/\sigma_x$, and Y) the condition that the universe from which the sample is drawn be normal is omitted, and there is little disposition to recognize that, especially for the higher moments, the expressions are derivable only on the assumption that the sample is very large. Indeed, neglect of this very important condition permits the author of Chapter VII to recommend a substitute for the χ test of goodness of fit which cannot be accepted unless this difficulty has been rigorously dealt with. Not only do the probable errors of his high α_s's mount up, as he points out, with increasing index, but, n being fixed, his formulas for them become increasingly uncertain. This is due to the fact that, strictly, there should be more terms in his formula for σ_{μ_s}, which involve higher powers of s. If n is fairly large, these terms are negligible when s is small, because they involve higher powers of $1/n$ than the terms retained, but when s is large they are very troublesome. There is the more need of insisting on the use of large samples because one of the common errors of practical workers in statistics is the error of using

in a more or less wooden way book formulas with inadequate data. A more grievous instance of the Handbook's failure to warn its readers against the use of small samples occurs in the chapter on multiple correlation, cited below.

There are a considerable number of places in the book where it seems to the reviewer that the statements should be modified or supplemented, and some of them will be noted as the several chapters are considered in turn. It is hoped that the reader will recognize, however, that the adverse criticisms are of details, and that, for the most part, they are offered with constructive purpose, rather than as indicative of general features. In fact, at first reading these details were seldom noticed. The general effect of the book was stimulating, and it was welcomed enthusiastically. Adverse criticism may probably be summarized in the statement that the book seems to have been written a bit too hastily. This book is such a useful contribution, and will occupy for some time to come such an important position in statistical literature, that it was well worth that extra effort which would have made that position permanent. It is much more than an elementary text, yet it just falls short of being a digest of a treatise, which one feels it might have been. The material is carefully selected and apportioned. The chapters differ markedly, but almost everywhere there is clearness, thoroughness, and simplicity of treatment, combined with mathematical exactness of statement which it is not usual to find in books on statistics. Numerical examples are used to illustrate almost every important idea.

The first chapter on "Mathematical Memoranda" comprises a collection of many mathematical formulas useful in statistics, and a very brief account of probability. The latter is rather too brief; at least it should include the point binomial theorem, and the probability theorem leading to the hypergeometric in finite form (Elderton, W. P., *Frequency Curves and Correlation*, p. 37). There are one or two mistakes noted later under "Errata." It should also be stated that the C's, at the bottom of page 16, are mutually exclusive. There should be a reference, at least in the bibliography, for the important table of Burgess, mentioned on page 14, viz., TRANSACTIONS OF THE ROYAL SOCIETY OF EDINBURGH, vol. 39 (1897), pp. 257–321. A valuable reference on quadrature formulas is Irwin, J. O., *Tracts for Computers*, No. 10 (1923). Irwin gives the constants of the Euler-Maclaurin theorem in two useful forms. On page 12, line 4, the words "absolute error" are used in a different sense from that in which they were defined on page 2.

Chapter II is an almost perfect introduction to the subject of frequency distributions. There is an obvious slip on page 32. The author should insert after several formulas, as he does on page 77, the words, "normal distribution". This restriction has an importance here, for on the next page he says "preference should be given to the average which has

the smallest probable error." Glancing back at page 32, the reader would infer that the arithmetic mean is therefore always to be preferred, but this it not a desirable inference; for on page 155 a case is cited where the distribution is not normal and the median appears to have the smaller probable error.

The third chapter deals with interpolation, summation, and graduation, and is clearly the work of one who has had much experience in this field. Considering the small space this chapter occupies, it is remarkably readable and comprehensive. The notation ${}_xC_k = x_k$ is confusing, since by u_k the author means the value of u at $x = k$. All his formulas are either successive averages or else based on interpolation by means of finite differences. As there are some functions to which the latter method does not well apply, a word of caution about the desirability of rapid convergence is needed. There is no mention of methods of interpolation on a surface (cf. Pearson, K., *Tracts for Computers*, No. 3, 1920). The author's formula for subdivision of an interval (p. 45) can be improved in form, provided the table on which it is to be used does not already contain its successive differences, and provided the computer is working with a machine which will subtract as well as it will add (Pearson, K., *Tracts for Computers*, No. 2, 1920). The author should give the dates in the references in his foot-notes. For example, at the bottom of page 45, it is misleading to cite the reference "Explanation of a New Formula for Interpolation" without the date, which is 1880!

Chapter IV on curve fitting is in Huntington's best style, simple, clear, and adequate. There follow two short chapters by Rietz, a thoroughly good account both of random sampling and of the distributions of Bernoulli, Poisson, and Lexis. The theorem attributed to Bernoulli on page 72 is not truly his. If it were, the normal law, which is implicit in it, and which the Handbook (following a bad custom) calls Gauss's, on page 11, would be Bernoulli's too. The *Ars Conjectandi* antedated Gauss by more than a century. This theorem is proved by Laplace, but subject to limitations which may make the second term worse than useless if c is not small. E. g., for the point binomial $(1/3 + 2/3)^{450}$, the approximation is better without the second term at $c = 3\sigma$. Somewhere in this book, probably in the chapter on sampling, there should be an explanation of the (different) formulas for probable errors commonly accepted by engineers, physicists, and astronomers (e. g., Chauvenet, *Practical Astronomy*, vol. 2, pp. 494, 506). The ordinary "theory of errors" is a special case of the theory of statistics, and is too frequently divorced from it. The desirability of severer warnings against the use of probable error formulas for small samples has already been mentioned. It is also desirable that there should be some warning against the usual *interpretation* of such formulas.

What is meant by the usual interpretation is indicated in the next chapter, on "frequency curves." At the top of page 100 there is a table giving the probability that a variable will differ in absolute value from its mean value by less than k times its probable error, provided the variable be normally distributed. Now, when the value of the probable error of a frequency constant has been determined, it is not surely correct to estimate by means of this table the probability that this constant will differ in absolute value from its mean value by less than k times its probable error. It is necessary to know first that this constant is normally distributed, and for this to be true it is not always sufficient to know that the distribution to which the constant belongs is normal. One may, of course, use the table to get an approximate estimate if one knows that the distribution of the constant is approximately normal, but this is frequently not true for small samples, as has been shown in the case of the coefficient of correlation by R. A. Fisher, (BIOMETRIKA, vol. 10 (1915), pp. 507–521). The author uses this table in connection with his high moment test of goodness of fit, to which an objection was made in an earlier paragraph. Waiving the question, there raised, as to whether, in a concrete case, he is finding the values of his probable errors correctly, it is now fair to raise the additional question: can he show that the distribution of his high a_s's is approximately normal? It is the opinion of the reviewer that the author's idea here, though interesting and original, is not as simple a matter as it appears, and that there was not space available in a handbook for the arguments which would be necessary to justify it. The author's account of Pearson's curves is an admirable simplification of the usual methods. His argument with regard to Sheppard's corrections (p. 94) is not very clear. Certainly the function $f(x)$ used in his analysis on the preceding page is *a* graduating function, and so the conclusion that Sheppard's corrections do not demand the vanishing of the "chosen" graduating function does not seem to follow necessarily from the analysis. The author admittedly has a point here, but it is a delicate one, and requires more careful handling. There should also be a warning against the use of these corrections in "abrupt" cases, where they sometimes would make the approximation worse.

Some mention has already been made of the two chapters following on simple and on multiple correlation. The first is a brilliantly clear chapter, perhaps the most striking in the book. The second suffers from an attempt to compress too much into its short space. If the space was necessarily restricted, it would seem to have been better to have attempted first to show what a solid of frequency is (which is not described at all), perhaps by the analogy of distribution of mass in a physical solid, to show what are meant, geometrically, by total, partial, and multiple regression, and by the other constants to be used, and

then merely to recite the formulas with numerical illustration. There appears, however, no good reason for such a strict compression of a subject which is well known to be difficult to present. The author does not always define his terms before using them. The term, "total correlation coefficient" is used on page 140, and first defined on page 141; "partial alienation coefficients" are defined in terms of "partial correlation coefficients" on page 141, but these in turn are first defined on page 142. The reader must be careful to note that the word "cell" is used in this chapter in a different sense from that in which it is used in the preceding chapter, and that the words "assigned values" and "measures" are used in the sense of observed values or measurements. The example chosen to illustrate the methods of computation contains only 136 data. This would not be objectionable if it were carefully pointed out that it is for purposes of numerical illustration only, but the serious discussion of the meaning of the results gives the impression that it is to be taken as an adequate and approved piece of statistical investigation in the field of education. This is extremely unfortunate, even though accidental, since it was hoped that the Handbook would be of special value in improving the quality of current statistical practice. Roughly speaking, it is as futile to attempt to find the parameters of a frequency solid of three dimensions from 136 data as it would be to seek the parameters of a frequency curve from $\sqrt[3]{136} = 5.14$ data. The situation is made worse in this case by the further fact that one of the variables does not represent a measurable character; there is only a division into three categories.

The last three chapters of the book are written by economic statisticians, on correlation in time series, periodogram analysis, and index numbers. They clearly set forth and illustrate methods that have been used by them, and will be a welcome addition to the earlier mathematical theory, especially in a country where the theory of statistics has been so largely identified with business.

Errata (partly supplied by the Editor). Page 16, line 12: *For* at least *read* some. Page 16, line 14: *For* at least one 2-spot with two dice *read* a 1-spot or a 2-spot with one die. Page 16, line 7 from bottom: *For* $\sqrt{2\pi npq}$ *read* $\sqrt{2npq}$. Page 54, line 5: *For* coefficients *read* coefficient. Page 56, line 5 from bottom: The minus sign (—) in $\frac{1}{2m}(u_0 - u_\infty)$ *should be changed to* $+$. Page 70, line 7 from bottom: *For* m^2 *read* m_1^2. Page 73, line 15: *For* 688 *read* 668. Page 74, line 6 of Ex. 2: *After* number *insert* of women's numbers; *to avoid ambiguity from the use of number in two senses.* Page 78, line 4 of second paragraph: *For* fit *read* fitted. Page 92: The integrals are definite and should have upper limits. Page 97, line 15: *For* $\frac{}{N}y$ *read* $\frac{\sigma}{N}y$. Page 103, line 7:

For 80 *read* 81. Page 105, line 3 from bottom: Omit x and the two commas following. Page 107, line 4: *For* V *read* VI *and for* VI *read* VII. Page 108, line 1: *For* VII *read* VIII. Page 121, line 4 from bottom: *For* $\frac{y_i'}{\sigma_y}$ *read* $\frac{y_i}{\sigma_y}$; *i. e., delete the prime.* Pape 126, formula (9): *For* $\sum_x^2$ *read* σ_x^2. Page 142: The last line should begin with $\beta_{12.34}$. Page 149, line 2: *For* (13) *read* (19). Page 187, formula (8): Insert Σ before parenthesis in the denominator.

B. H. CAMP

FOUR BOOKS ON PROBABILITIES

Éléments de la Théorie des Probabilités. Third edition. By Emile Borel. Paris, J. Hermann, 1924. vii + 266 pp.

Probabilités, Erreurs. By Emile Borel and Robert Deltheil. Paris, Librairie Armand Colin, 1923. vi + 197 pp.

Wahrscheinlichkeitsrechnung. By Otto Knopf. Sammlung Göschen. Berlin, Walter de Gruyter and Co., 1923. Two volumes, 112 and 112 pp.

Grundlagen der Wahrscheinlichkeitsrechnung und der Theorie der Beobachtungsfehler. By F. M. Urban. Leipzig, B. G. Teubner, 1923. vi + 274 pp.

The present time is a time of decided activity in the publication of books on the theory of probability. While two or three works of considerable originality and merit have appeared, the majority are merely text-book rearrangements of the traditional course in probability adapted to various classes of readers.

The first of the books in this review is the third edition of Borel's *Théorie des Probabilités.* This well known work has been before the public since 1909. The chief change in this third edition is the addition of four notes at the end of the book on applications to radio-activity, on a problem leading to a Stieltjes integral, on games of chance in which the ability of the players is taken into consideration, and on what Borel calls a differential method in statistics.

The book by Borel and Deltheil is number 34 of the series of little books known as the "Collection Armand Colin". The purpose of the series is to present to the educated person readable monographs on special topics in philosophy, history, science, mathematics, literature and other branches of learning. As might be expected, the book covers much the same ground as the more extended work of Borel.

It emphasizes the applications to statistics, biology and physics. The last chapter is on the method of least squares and the theory of errors from the Gaussian standpoint. A number of interesting exercises and problems, including some of the well known problems in the theory of probability, are given at the ends of the chapters. The work is a very readable little book for the non-specialist, although the reader must have some knowledge of calculus.

The third work is one of the series of books well known as the SAMMLUNG GÖSCHEN. Two volumes of the series are used. This work covers the traditional topics in the orthodox formal manner of many English and German text books. It seems to replace the one volume work on probability by Hack in the SAMMLUNG GÖSCHEN.

The work by Urban is an attempt to present to students of the sciences the fundamentals of the theory of probability. The book has rather a curious history. It was originally written in English while the author was connected with the faculty of the University of Pennsylvania, and the intention was to publish in the United States a rather complete treatise on the theory of probability. The war intervened and the author returned to Germany and published in German this abridged work emphasizing the underlying principles and the theory of errors. This was a distinct loss to the English reading student. It still seems easier to find a publisher in Germany than in the United States. The reviewer feels that the more complete book would have been much more acceptable than the part finally published. The book is not like a formal text book on the theory of probability. It is the aim of the author to make clear the fundamental notions underlying his subject, and in doing so he uses the "wordy" method, using the term in a good sense. He is not always successful in this clearing process, especially when one considers the layreader, but he does thoroughly discuss the debatable topics in probability from all points of view. For example, he devotes many pages to the word *Wahrscheinlichkeit*, and reviews the definitions, views, objections and remarks of various writers on this fundamental word. It is an interesting and valuable discussion but many objections could be raised to the author's insistence upon his Mengenlehre definition of probability, especially when the non-technical student is considered. The chapter on the theory of errors is unique. It is not the type of chapter which tells the student how to handle errors of observation but one which discusses the hypotheses and notions at the foundation of the theory in an informal, open-minded manner.

A. R. CRATHORNE

SHORTER NOTICES

Geschichte der Elementarmathematik. By Johannes Tropfke. Berlin and Leipzig, Walter de Gruyter. Bd. V: *Ebene Trigonometrie. Sphaerik und Sphaerische Trigonometrie.* i+185 pp. 1923. Bd. VI: *Analysis. Analytische Geometrie.* i+169 pp. 1924.

The fifth and sixth volumes of the revision of Tropfke's history of elementary mathematics maintain the high standard set by the first four volumes*. Here is the same wealth of well-arranged material, the same concise and yet vivid style, the same care in evaluating all the contributions of previous workers in the field. The revision includes large amplifications, as the topics contained in these two volumes of 354 pages were covered in 251 pages of the same size in the first edition. The number of references to the literature in the form of footnotes has been increased from 971 to 1922.

The topics are: Vol. 5, plane trigonometry, pp. 3–98; spherical geometry and trigonometry, pp. 101–185; Vol. 6, series, pp. 3–55; compound interest, pp. 56–62; permutations and probability, pp. 63–74; continued fractions, pp. 74–84; maxima and minima (in elementary geometry), pp. 84–91; analytic geometry, pp. 92–169.

The discussion of analytic geometry is a notable example of the improvement introduced in the new edition. The algebraic-geometric problems of al-Khowarizmi and Abu Kamil are given in some detail; the descriptions of Descartes's *Géométrie* and of Fermat's *Isagoge* have been made somewhat fuller and clearer; and the influence of Descartes's work upon his contemporaries and successors has been traced in a more satisfactory manner.

A detail that is not without general interest is in reference to "Heron's formula" for the area of a plane triangle in terms of its sides: $F = \sqrt{s(s-a)(s-b)(s-c)}$. In the revised edition, Tropfke accepts the statement of an Arab writer of the 11th century† that this formula is not original with Heron, but is due to Archimedes. Heath in his *History of Greek Mathematics*‡ also accepts this statement. In spite, however, of these excellent precedents, and of the fact that there is obviously nothing inherently improbable in the ascription to Archimedes, the reviewer would prefer to await more conclusive evidence before rejecting the tradition which is supported by so many ancient writers, because of this one contrary testimony.

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* Reviewed in this BULLETIN, Vol. 29 (1923), pp. 476–477.

† BIBLIOTHECA MATHEMATICA (3), vol. 11 (1910–11), p. 39.

‡ Vol. 2, p. 322.

There are a few misprints; in volume 5, p. 61, lines 15 and 16, sin 2α and cos 2α schould be sin α and cos α respectively; p. 74, line 4 from end, Gegenseiten should read Gegenwinkel; and p. 85, line 16, 1675 should be 1765.

The seventh and final volume is to contain Stereometry and a complete index. The appearance of this index will be impatiently awaited, as it will increase the value of the set many fold. Even as it is, the six volumes are indispensable for the teacher or student of the history of elementary mathematics.

R. B. McClenon

Principles of Geoemtry. By H. F. Baker. Vol. II: *Plane Geometry, Circles, Non-Euclidean Geometry.* Cambridge University Press, 1922. xv + 243 pp.

This book, in continuation of the first volume, aims to present the main theorems of plane geometry and to develop logically the results of the principles explained in the first volume. In both purposes the author has succeeded admirably.

The preliminary chapter of the present text reviews in a brief manner enough of the matter of the first volume to enable a reader to use the present volume without reference to the first one, provided he has an elementary knowledge of projective geometry. In both volumes the treatment is first synthetic. The fundamental notation is that of projective, or (as the author calls them) related ranges. The notions of distance and congruence are not assumed. These notions and coordinate systems are developed later with a study of the logical principles underlying them.

In chapter one of the present volume, the general properties of conics are deduced from their definition as the locus of the intersection of the corresponding rays of two projective pencils and a wealth of theorems are presented.

In chapter two the relation of geometric figures to two given points of reference are studied. Let us assume any two points I and J as the absolute. Then if a line AB meets the line IJ in a point K, the point C which is the harmonic conjugate of K with respect to A and B is the midpoint of AB. Two lines which meet on IJ are parallel, and two lines are perpendicular if they meet IJ in two points which are harmonic conjugates with respect to I and J. A circle is a conic through I and J. From these definitions the usual properties of circles are deduced and a discussion of coaxial circles, inversion on a circle and the like are given. Similarly, projective definitions of foci of a conic, of a rectangular hyperbola, of a parabola and the like may be given, and the so-called metrical properties of conics obtained.

In chapter three, under the heading of the equation of a line and of a conic, a coordinate system is for the first time introduced. From the author's viewpoint the use of symbols is unnecessary but is convenient as rendering a demonstration easier to follow. If A, B, C are symbols attached to any three points of a plane, not lying in the same straight line, then any other point P in the plane has a symbol of the form $xA+yB+zC$ where x, y, z are algebraic quantities. This has been shown in volume one as a consequence of postulates of incidence and the theorem of Pappus. Then (x, y, z) or (mx, my, mz) are the coordinates of P. With the coordinates thus introduced the usual analytic geometry is readily developed.

In chapter four there is discussion of the use of imaginary elements in geometry and the logical questions involved.

Chapter five treats of projective measurement of distance and angle with reference to a fundamental conic, and discusses somewhat fully the types of non-euclidean geometry which result.

F. S. WOODS

Mathematical Theory of Life Insurance. By C. H. Forsyth. New York, John Wiley and Sons, 1924. 6 + 74 pp.

Books dealing with the mathematics of finance and life insurance may be divided roughly into two classes: (1) those that present thousands of formulas; (2) those that attempt to select a comparatively few as fundamental. While certain students may learn best from the first type of book, the second type is more attractive and more inspiring. Forsyth's text is of the second type; and in the reviewer's opinion, the selection of topics and formulas to be stressed is decidedly felicitous. The first half of the book presents with great simplicity and direct nessthe rudiments of probability, the mortality table, and the premiums for annuities and insurances. As an analogue for ${}_nE_x$, Forsyth introduces the symbol ${}_nI_x$ as the single premium for insurance covering the nth year only; and thus $A_x = \sum {}_nI_x$, just as $a_x = \sum {}_nE_x$. Chapter IV explains rather briefly but comprehensively the valuation of policies in accordance with the laws of different states, and this chapter is decidedly important. The instructor will have to supplement this to some extent, and may prefer to give the prospective method before the retrospective method. In using the "shuttle" ${}_nE_x$, it should be noted that the "x" refers to the earlier age, not to the age from which the change is made. Chapter V explains in a simple manner the Makeham formula and its use in joint insurance, leaving for the two-page appendix the derivation of the Makeham formula. The American Experience Table is given, with columns for l_x, d_x, q_x, p_x, e_x, D_x, N_x, M_x, the three latter on the basis of $3\frac{1}{2}$ %. Numerous exercises appear throughout the text.

E. L. DODD

Lehrbuch der Algebra, verfasst mit Benutzung von Heinrich Webers gleichnamigem Buche. Erster Band: *Allgemeine Theorie der algebraischen Gleichungen.* By Robert Fricke. Braunschweig, F. Vieweg und Sohn, 1924. viii + 468 pp.

When the last edition of the *Lehrbuch der Algebra* by the late H. Weber became out of print, the publishers invited Professor Fricke to write a treatise on algebra in three volumes to replace that by Weber. The subject matter of the present volume 1 is essentially the same as Weber's volume 1 and the earlier chapters on abstract groups in Weber's volume 2. Fricke states that his second and third volumes will differ more essentially from those by Weber. Volume 2 will treat those algebraic equations which are not solvable by radicals and possess groups representable by binary and ternary substitutions; in particular, Klein's theory of the icosahedron and equations of the fifth degree will be presented from the geometrical standpoint of Klein (in contrast to Weber's algebraic treatment). Volume 3 will develop the theory of algebraic numbers and the class equation of complex multiplication of elliptic functions.

Professor Fricke is an experienced writer of unusual clearness. In addition to his recent book on elliptic functions, he was joint author with Klein of extended treatises on elliptic modulur functions and automorphic functions. These earlier books show that the author has long been familiar with the field covered by the new algebra. The latter will be somewhat simpler to read than Weber's algebra. General determinants are given a better notation, and their rank is employed systematically, thus avoiding the circumlocution of Weber. Certain proofs are replaced by simpler or more natural ones; for example, the product of two determinants, and the determination of the number of positive signs in the canonical form $\sum \pm x_i^2$ of a quadratic form.

There are brief treatments of the following topics omitted by Weber: k-fold bordered determinants, linear dependence, rank of symmetric matrices, besides the additions on pages 6–7, 10–13, 67.

Certain topics in Weber are omitted by Fricke: elimination of two unknowns from three equations; domains defined by an imaginary cube root or fourth root of unity; most of Weber's chapter 11 on continued fractions (perhaps those domains and the theory of the equivalence of quadratic numbers will be treated in volume 3).

Although the general theory of transformations of Tschirnhaus is developed at length, there is no mention of the chief application to the reduction of the general equation of the fifth degree to a normal form involving a single parameter, whether the Bring-Jerrard form $y^5 + y + c = 0$ or the still more important Brioschi form. Doubtless these normal forms will be given in volume 2. But it is far preferable that the reader should meet them as the culmination of the elaborate theory of transformations of Tschirnhaus (Weber I, 2d ed., p. 205, p. 263)

Since the present volume covers nearly all the ground of Weber's volume 1 of 703 pages, as well as the first two chapters (of 68 pages) on abstract groups in Weber's volume 2, it may seem strange that Fricke's present book contains only 461 pages, besides the index. But Fricke's pages are larger than Weber's and the type is materially smaller.

The owner of a copy of Weber's algebra who uses it chiefly as a reference book will not find it essential to buy Fricke's book. But a person who expects to do considerable reading in one of these books will find it to his advantage to own the simpler book by Fricke. The advantage in favor of Fricke's volumes 2 and 3 will doubtless be relatively greater than in the case of volume 1.

It is a pity we do not have in English a treatise on algebra as good as either Weber's or Fricke's.

L. E. DICKSON

Leçons de Géométrie Élémentaire. By Édouard A. Fouët. Paris, Vuibert, 1924. xvi + 348 pp.

This is an elementary geometry for pupils who are familiar with theorems *les plus simples* on the straight line and circle. It consist of eight books: I and II on the straight line and circle, III on similar figures, IV on plane areas, V on the plane and line, VI on polyhedra, VII on the sphere, cylinder, and cone, VIII on the conic sections; together with a preliminary book (to be read last, or at least later), a résumé of definitions, transformations, loci, and methods for solution of problems. Intermingled in each book are paragraphs of three distinct natures, (1) an ordinary course in geometry, (2) a course in problems, (3) an excellent set of pedagogical and historical notes.

The ordinary course (1) needs only the obvious commentary that like most similar Continental works it is incomparably better than American geometries in the development of broad underlying principles. In appropriateness of application and in development of power the problem course (2) compares favorably with even such a specialized book as Petersen's *Methods and Theories for the Solution of Problems of Geometrical Construction*, while in wealth of advanced material it is only slightly inferior to Hadamard's *Leçons de Géométrie Élémentaire*. A superior book; but demanding superior teaching, for the course (1) must be accompanied by not too difficult problems (of which there is perhaps a scarcity), and those demanding more maturity must be left until later. But still these last are there, forming a part of the text, not grouped at the end of the chapter or book. This is perhaps the outstanding feature of this text.

B. H. BROWN

An Introduction to the Mathematical Analysis of Statistics. By C. H. Forsyth. New York, John Wiley and Sons, 1924. 8 + 241 pp.

The preface advises us: "This book is offered as a textbook in mathematics and not as a reference book for the statistician". A knowledge of calculus is assumed, but the text lies strictly within the grasp of an undergraduate. Before taking up the usual topics of a book on statistics, a substantial mathematical background is established — as may be seen from the chapter headings: Errors and numerical computation; Finite differences; Interpolation; Gamma and beta functions; Probability; Averages and aids in their computation; Moments; The normal curve; The binomial $(p+q)^n$. Statistical series; Correlation theory. Few texts on statistics have been written in English which have such an admirable selection of topics for a junior course in mathematics. Without going into intricate details, the book is decidedly comprehensive, and introduces the student to a multitude of problems in statistical research. It is written in an attractive style — quite informal — and on this account an instructor will need to supplement some statements, by using more technical phraseology. For example, the student should be put on notice that "compensating errors" (p. 3) do not compensate, and that in tangential interpolation (p. 42) the continuity of the curve is not involved. Moreover, for (27) on page 57 $a > 0$; the illustration on page 81 is not felicitous; normality (p. 93) involves a great deal more than symmetry; the "set of observations" at the foot of page 94 is a set of deviations; the bisection of area (p. 119) is effected by the ordinate through the *median*; the argument (p. 139) for the differential equation of the normal curve is hardly convincing. Besides the useful 6-place tables of logarithms and antilogarithms at the back of the book, there are some half-dozen other tables that might well be placed at the back.

E. L. Dodd

Methodik des mathematischen Unterrichts. Dritter Teil: *Didaktik der angewandten Mathematik.* By W. Lietzmann. Leipzig, Quelle und Mayer, 1923. xi + 234 pp.

This is the third and last volume of a work of which the second volume has already been reviewed in this Bulletin, (1923, p. 479). The general characteristics of the present volume on "Applied Mathematics" are similar to those of its predecessor. Accordingly, it will suffice to mention here the various fields whose mathematical aspects are discussed in so far as they fall within the scope of secondary school mathematics, namely: linear drawing, manual training, geodesy and astronomy, commerce and politics, mechanics, physics, philosophy.

J. W. A. Young

NOTES

At the Summer Meeting of the Society at Cornell University, on September 8–12, the Tenth Colloquium of the Society will consist of two courses of lectures: *The new differential geometry*, by Professor L. P. Eisenhart; and *The theory of approximation*, by Professor Dunham Jackson.

The Western Christmas Meeting of the Society and the meeting of the Southwestern Section of the Society will be held at Kansas City, in conjunction with the meeting of the American Association for the Advancement of Science. At the time of these meetings, Professor James Pierpont will deliver the annual Gibbs lecture, on the topic *Some modern views of space*. It is planned to hold the ordinary meeting on Tuesday, December 29, to have a joint session with the Mathematical Association and with Section A on Wednesday morning, and to have the Gibbs lecture presented before a general session of the American Association on Wednesday afternoon. The ordinary meeting of the Mathematical Association will probably fall on Thursday.

The opening number of volume 27 of the TRANSACTIONS OF THIS SOCIETY (January, 1925) contains the following papers: *On normal forms of differential equations*, by W. F. Osgood; *Congruences with constant absolute invariants*, by H. L. Olson; *On the prime divisors of the cyclotomic functions*, by C. M. Huber; *On the roots of the Riemann zeta function*, by J. I. Hutchinson; *A generalization of the Riemannian line element*, by J. L. Synge; *Elementary functions and their inverses*, by J. F. Ritt; *Analytic transformations of everywhere dense point sets*, by Philip Franklin; *An algebraic solution of the Einstein equations*, by Edward Kasner; *Electrodynamics in the general relativity theory*, by G. Y. Rainich.

At the meeting of the Division of Physical Sciences of the National Research Council in Washington on April 26, the following officers were elected for the year beginning June, 1925: Chairman, J. S. Ames; Secretary, W. E. Tisdale; Vice-Chairman, F. K. Richtmyer; Executive Committee, J. S. Ames (chairman), G. D. Birkhoff, P. D. Foote, D. C. Miller, L. Page, F. K. Richtmyer. The following persons were elected members at large: G. D. Birkhoff, D. L. Webster. New members representing scientific societies were announced as follows: for the American Astronomical Society, J. Stebbins; for the American Mathematical Society, L. P. Eisenhart; for the American Physical Society, A. J. Dempster and A. L. Foley; for the Division of Federal Relations, W. Bowie.

The British Association for the Advancement of Science will meet at Southampton, August 26–September 2, 1925. Dr. C. G. Simpson has been appointed President of Section A (mathematics and physics) and Dr. A. M. Tyndall recorder of that Section for this meeting.

The Westphalian Mathematical Society announces that it will hold a celebration in honor of Weierstrass (Weierstrass-Woche) from June 2 to June 6, 1925, at the University of Münster. Addresses will be made by G. Mittag-Leffler, D. Hilbert, O. Perron, P. Koebe, L. Bieberbach, and K. Knopp.

The Cambridge University Press has in preparation a new edition of Whitehead and Russell's *Principia Mathematica*, of which the first volume will appear shortly.

The Royal Belgian Academy has awarded its Decennial Prize for Applied Mathematics for the period 1913–1922 to Professor T. De Donder, for his works on the Einstein theory of gravitation and his *Leçons de Thermodynamique et de Chimie Physique.*

At the centenary celebration in Copenhagen of the Society for the Spread of Natural Philosophy, the Oersted gold medal of that Society was presented to Professor Neils Bohr.

The Alfred Ackermann-Teubner Memorial Prize for the promotion of mathematical sciences has been awarded to Dr. Arnold Kohlschütter, of the Astrophysical Observatory at Potsdam, for his determination of the absolute brightness of stars from the intensity relations of certain spectral lines announced in volume 40 of the ASTROPHYSICAL JOURNAL.

The Ernst Abbe Prize, founded in 1921 by Carl Zeiss for the furtherance of mathematics and physics, has been awarded to Professor Felix Klein, for his mathematical work.

Sir J. J. Thomson has been awarded the Faraday medal of the Institution of Electrical Engineers, London.

The gold medal of the Royal Astronomical Society has been awarded to Sir Frank Dyson, Astronomer Royal, for his contributions to astronomy, and in particular for his researches on the proper motions of the stars.

The Carnegie Institution has granted to Professor D. N. Lehmer, of the University of California, a preliminary sum of $ 500 with which to start the work of constructing the stencils described in his paper, *On a new method of factorization* (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 11, No. 1).

Among the grants in aid of research announced by the American Association for the Advancement of Science for 1925 are the following:

$250 to Dr. H. W. Stager, of the University of Washington, for building a computing machine for use in constructing a list of prime numbers and a factor table; $300 to Professor A. O. Leuschner, of the University of California, for use in the study of the perturbations of minor planets.

Professor J. Hadamard, of the Collège de France, arrived in this country on April 26 and returned to France on June 17. He spent this period of two months in travel and in lecturing at the universities of Illinois, Wisconsin, Indiana, California Southern Branch, Stanford, California (Berkeley), and Rice Institute. The subjects of some of his lectures were: *Huyghens' principle, Some recent points of view on the notion of function. The second period of the work of Poincaré on differential equations, and Some recent improvements in the geometry of circles in space.*

On the occasion of the Third Pan-American Scientific Congress (see this BULLETIN, vol. 30, p. 566, and p. 290 of this number), the University of San Marcos, at Lima, conferred the degree of Honorary Doctor in the Faculty of Sciences on Professor E. V. Huntington, of Harvard University.

At the University of Catania, Dr. G. Albanese, Dr. G. Andreoli, and Dr. A. Terracini have been appointed to associate professorships, in the fields of projective and descriptive geometry, the calculus, and analytic geometry, respectively.

At the University of Cagliari, Dr. B. Caldonazzo has been appointed associate professor of rational mechanics.

At the University of Florence, Dr. F. Tricomi has been appointed associate professor of analysis.

At the University of Genoa, Dr. P. Straneo has been appointed associate professor of mathematical physics.

Professor Gustav Herglotz, of the University of Leipzig, has been appointed professor of mathematics at the University of Göttingen.

Professor B. Janet, of the University of Rennes, has been appointed professor of differential and integral calculus at the University of Caen, as successor to Professor C. Riquier, retired.

At the University of Messina, Dr. U. Crudeli has been appointed associate professor of mathematical physics.

Dr. L. Neder has been appointed to an associate professorship at the University of Leipzig.

Professor G. Pólya, of the Zurich Technical School, has been awarded a Rockefeller travelling fellowship for a year's residence at Oxford and Cambridge.

Professor H. Tietze, of the University of Erlangen, has been appointed to a professorship at the University of Munich.

Professor F. Meyer, of Königsberg, and Professor C. Runge, of Göttingen, have retired from active teaching.

The following have been admitted as privat docents: Dr. H. Behnke, at the University of Hamburg; Dr. F. Krauss, at the Technical School at Aachen; Dr. A. Walther, at the University of Göttingen; Dr. J. Wellstein, at the Technical School at Karlsruhe.

The following appointments as privat-docents in Italian universities are announced: University of Florence, Dr. E. Fermi; University of Palermo, Mrs. Margherita Piazzolla Beloch.

Professor A. A. Bennett, of the University of Texas, has been appointed head of the department of mathematics at Lehigh University.

At the University of North Carolina, Associate Professors A. W. Hobbs and J. W. Lasley, Jr., have been promoted to full professorships and Assistant Professor A. S. Winsor to an associate professorship of mathematics; Mr. J. B. Linker has been promoted to an assistant professorship of applied mathematics.

Assistant Professor V. B. Hinsch, of the Missouri School of Mines, has been promoted to an associate professorship of mathematics.

Assistant Professor H. M. Morse, of Cornell University, has been appointed associate professor of mathematics at Brown University.

Dr. H. L. Olson, of the University of Michigan, has been appointed assistant professor of mathematics at Michigan Agricultural College.

Professor G. E. F. Sherwood, of the Southern Branch of the University of California, has been granted leave of absence for the next academic year, and will go to Cambridge, England, for study.

Mr. Telesforo Tienzo has been promoted to an assistant professorship of mathematics at the University of the Philippines.

The following appointments to instructorships are announced:
Columbia University, Dr. M. H. Stone;
University of Michigan, L. M. Blumenthal, H. F. Schiefer, E. H. Wagner;
University of North Carolina, Mr. C. H. Benson;
Randolph-Macon College, S. T. Arnold and W. S. McClintic.

Professor Paul Urysohn, of Moscow, died August 17, 1924.

Professor P. Schafheitlin, of the Berlin Technical School, died October 2, 1924.

Professor G. J. Wallenberg, of the Berlin Technical School, died November 16, 1924, at the age of sixty years.

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